September 2, 2022

Dear Andres I. N.,

I am writing to include some of the things that we discussed today, so that I don't forget.

The main thing that we were left with was the following. Everything is Noetherian, I always assume diagonals are quasicompact.

Proposition 0.1. *Let $\mathfrak{X} \to S$ and $\mathfrak{Y} \to S$ be (algebraic stack) gerbes over an algebraic space S. Suppose that $\mathfrak{Y}$ has relative affine diagonal. If $\pi : \mathfrak{X} \to \mathfrak{Y}$ is a good moduli space morphism, then π is also a gerbe.*

Proof. By [Sta22, Tag 06QH], there is an fffp morphism $U \to S$ of schemes and an fppf U-group algebraic space G such that the stack $B_U G := U/G$ fits into a cartesian diagram

$$\begin{array}{ccc} BG & \longrightarrow & \mathfrak{X} \\ \downarrow & & \downarrow \\ U & \longrightarrow & S \end{array}$$

A similar statement holds for $\mathfrak{Y}$ for some ffp cover $V \to S$. By taking the common refinement $U \times_S V \to S$, we can assume that $V = U$. Since we can check that $\mathfrak{X} \to \mathfrak{Y}$ is a gerbe fppf locally on the base S [Sta22, Tag 06QF], and since good moduli spaces are preserved by base-change of base algebraic spaces, we can replace S by U and assume that $\mathfrak{X} \cong B_U G$ and $\mathfrak{Y} \cong B_U H$ for some fppf U-group algerbraic spaces G and H.

The morphism $B_U G \to B_U H$ corresponds to a homomorphism of U-groups schemes $\varphi : G \to H$. By Lemma 0.4, it suffices to show that the morphism $G \to H$ is surjective and flat. By the fiberwise criterion for flatness [Sta22, Tag 039E], this can be checked on geometric fibers of U, and so we can assume without loss of generality that $U = Spec(k)$ for some algebraically closed field k. In that case G, H are algebraic groups of finite type over k. Let $K \subset G$ denote the kernel of the homomorphism $G \to H$. Then we can factor $G \to G/K \hookrightarrow H$. We want to show that $G/K = H$. We can factor

$$BG \to B(G/K) \to BH$$

The first morphism $BG \to B(G/K)$ is a gerbe by Lemma 0.4, and the relative automorphisms of the unique k-point are isomorphic to K. We can base-change the morphisms $BG \to B(G/K) \to BH$ by the tivial flat cover $Spec(k) \to BH$ in order to obtain $H/G \to H/(G/K) \to Spec(k)$, where $H/G \to H/(G/K)$ is a gerbe and $H/(G/K)$ is a scheme. The automorphisms of any k-point in H/G are isomorphic to K. Since $H/G \to Spec(k)$ is a good moduli space, we conclude that K is linearly reductive. Therefore the gerbe $H/G \to H/(G/K)$ is also a good moduli space. By uniqueness of good moduli spaces, we conclude that $H/(G/K) \cong Spec(k)$, which implies that $H = G/K$, as desired. □

Remark 0.2. *Instead of using algebraic spaces, we can use the left cancellation property of good moduli space morphisms. Let $\mathfrak{X} \to \mathfrak{Y} \to \mathfrak{Z}$ be morphisms of algebraic stacks.*

If both $\mathfrak{X} \to \mathfrak{Y}$ and $\mathfrak{X} \to \mathfrak{Z}$ are good moduli space morphisms, then $\mathfrak{Y} \to \mathfrak{Z}$ is a good moduli space morphism. This implies in the argument above that $H/(G/K) \to Spec(k)$ is a good moduli space morphism, which necessarily means that $H/(G/K) \cong Spec(k)$, because cohomologically affine morphisms of schemes are affine.

Remark 0.3. *The proposition is not true if we don't require the diagonal of $\mathfrak{Y}$ to be affine. For example let k be a field, and let A be an abelian variety over k. Then the morphism $Spec(k) \to BA$ is a good moduli space morphism, but it is not a gerbe.*

Lemma 0.4. *Let $\varphi : G \to H$ be a homomorphism of flat finitely presented groups schemes over some scheme U. Then the corresponding morphism $B_U G \to B_U H$ is a gerbe if and only if φ is flat and surjective.*

Proof. Suppose first that $\varphi : G \to H$ is flat and surjective. Let K denote the kernel of φ, so that $K \hookrightarrow G$ is a locally closed subgroup scheme. The cartesian diagram

$$\begin{array}{ccc} K \times G & \longrightarrow & G \\ \downarrow & & \downarrow \\ G & \longrightarrow & H \end{array}$$

shows that $K \times G \to G$ is flat. Since $G \to U$ is flat, it follows that $K \to U$ is a flat group scheme. In particular, the algebraic space quotient G/K makes sense (it is the fppf sheafification of the naive quotient functor). Moreover, since $K \hookrightarrow G$ is a normal subgroup scheme and G is U-flat, it follows that G/K is a U-flat group algebraic space. The K-invariant morphism $G \to H$ induces a homomorphism $G/K \to H$. After base-changing via the flat $G \to H$, this becomes an isomorphism $G \xrightarrow{\sim} G$, and so $G/K \cong H$. Now, after base-changing by the trivial flat cover $U \to B_U H$, the morphism $B_U G \to B_U(G/K) \cong B_U H$ becomes the gerbe $B_U K \to U$. Therefore, by [Sta22, Tag 06QF], it follows that $B_U G \to B_U H$ is a gerbe.

Conversely, suppose that $B_U G \to B_U H$ is a gerbe. We want to check that $\varphi : G \to H$ is flat and surjective. (Thank you for suggesting the following diagram) Consider the fiber product diagram

$$\begin{array}{ccc} G & \longrightarrow & H \\ \downarrow & & \downarrow \nu \\ B_U G & \xrightarrow{\Delta} & B_U G \times_{B_U H} B_U G \end{array}$$

The stack $B_U G \times_{B_U H} B_U G$ classifies triples $(\mathcal{P}_1, \mathcal{P}_2, \psi)$, where $\mathcal{P}_1, \mathcal{P}_2$ are G-bundles and ψ is an isomorphism between the corresponding associated H-bundles. The morphism $\nu : H \to B_U G \times_{B_U H} B_U G$ corresponds to the two trivial G-bundles $\mathcal{P}_1, \mathcal{P}_2$ on H, and the isomorphism of the corresponding trivial associated H-bundles given by the universal point $\psi \in H(H)$ determined by the identity morphism $H \to H$. Since $B_U G \to B_U H$ is a gerbe, the relative diagonal $\Delta : B_U G \to B_U G \times_{B_U H} b_U G$ is flat and surjective [Sta22, Tag 0CPS]. By the Cartesian diagram above, it follows that $G \to H$ is flat and surjective, as desired. □

Remark 0.5. *(Alternative old argument for the fact that $B_U G \to B_U H$ is a gerbe $\implies$ $G \to H$ flat and surjective.)*

Suppose that $B_U G \to B_U H$ is a gerbe. We want to check that $\varphi : G \to H$ is flat and surjective. By the fiberwise criterion for flatness [Sta22, Tag 039E], we can check this over geometric points of U, and so we can assume without loss of generality that $U = Spec(k)$ for an algebraically closed field k. Then we can factor $G \to G/K \hookrightarrow H$, where K is the kernel of φ. We want to show that $G/K = H$. Factor $BG \to B(G/K) \to BH$. By base-changing to the trivial flat cover $Spec(k) \to BH$, we obtain the diagram $H/G \xrightarrow{p} H/(G/K) \xrightarrow{q} Spec(k)$. Here $H/G \to Spec(k)$ is a gerbe by assumption, and $H/G \to H/(G/K)$ can be plainly seen to be a gerbe (alternatively use the converse direction). Therefore both are universal homeomorphisms, and we have $p_*(\mathcal{O}_{H/G}) = \mathcal{O}_{H/(G/K)}$ and $(q \circ p)_*(\mathcal{O}_{H/G}) = k$. It follows that the representable morphism $H/(G/K) \to Spec(k)$ is a universal homeomorphism. This implies that $H/(G/K) \xrightarrow{q} Spec(k)$ is affine (the scheme $H/(G/K)$ must consist of a single point!). We also have that $q_*(\mathcal{O}_{H/(G/K)}) = q_*(p_*(\mathcal{O}_{H/G})) = k$, and so the affine morphism $H/(G/K) \xrightarrow{q} Spec(k)$ must be an isomorphism, as desired.

Best regards,
Andres

References

[Sta22] The Stacks Project Authors. *Stacks Project*. https://stacks.math.columbia.edu, 2022.