November 28, 2022

Dear Andres I. N.,

I am sketching the proof we discussed so that I have it on record.

Proposition 0.1. *Let S be a locally Noetherian reduced scheme. Let $H \to G$ be an isomorphism of affine group schemes over S. Suppose that $H \cong \mathbb{G}_m^h$ for some integer $h \geq 0$. Then, the kernel of φ is flat over S, and is a diagonalizable groups scheme over S.*

Proof. By the valuative criterion for flatness, we can reduce to the case when the base is a $Spec(R)$ for a discrete valuation ring R. The group H acts on H by translation (using the homomorphism φ). More concretely, t the level of rings, we have that $R[G] \to R[H]$ is a morphism of graded rings (graded by the free group $X^*(H)$ of characters of H). Let Z be the scheme theoretic image of $\mathbb{H} \to G$. This also acquires an action of H (the coordinate ring of Z is the image of $R[G]$ inside $R[H]$.

Let S be a ring over R, and let $p : Spec(S) \to H$ be an S-point of H. Then p is an S-point of the kernel iff it acts trivially on the H_S module $R[Z] \otimes_R S$. This follows by tracing definitions, because the Z contains the identity section of G, and the S-point is trivial iff it fixes the identity section.

Crucial point: The coordinate ring $R[Z]$ is a free R module! First $R[Z]$ is projective because it is contained in the free R-module $R[H]$, and R is a hereditary ring. This implies that $R[Z]$ is free, because every projective module over a local ring is free.

Decompose $R[Z] = \bigoplus_{\chi \in X^*(H)} R[Z]_\chi$ in terms of H-weights. Each graded piece $R[Z]_\chi$ is also a free R-module. So for any R-algebra S, the H_S-weights appearing in the decomposition of $R[Z] \otimes_R S$ remain always the same, they are just the set of weights χ such that $R[Z]_\chi \neq 0$.

A moment of thought shows then that the kernel is isomorphic to the intersection $\bigcap_{\chi \text{st} R[Z]_\chi \neq 0} \mathrm{Ker}(\chi)$ of the kernels of the weights appearing in $R[Z]$. You can replace this possibly infinite set of weights with a set of generators of the subgroup $\Sigma \subset X^*(H)$ spanned by them. Then the intersections of the kernels is obviously flat, as desired. □

Remark 0.2. *You mentioned that Dan showed that the weights spaces of the pointwise kernels of any morphism $\mathbb{G}_m^h \to G$ are locally constant, where G is an arbitary group scheme of finite type. I think that this might show that the same result holds true keeping the assumption that the base S is reduced, but removing the assumption on the affiness of G.*

Remark 0.3. *As I outlined during our conversation, the flatness of the kernel holds over any base (possibly nonreduced) if G admits a monomorphims to $GL_n \times S$ for some n.*

Question: It is natural to wonder whether this might hold more generally removing the reducedness conditions of S. I thought for a bit and did not get a counterexample. But there could very well be a simple one.

Best regards,

Andres