Hartogs's property for BG

Andres Fernandez Herrero

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Proposition 1 (Hartogs property)**.** *Let G be an affine smooth geometrically reductive group scheme over Noetherian base scheme S. Let Y be a regular scheme equipped with a morphism to S, and let $U \subset Y$ be an open subscheme of Y such that every point of the complement has codimension 2 in Y. Then, for any morphism $f : U \to BG$ there is an extension $\widetilde{f} : Y \to BG$ that is unique up to unique isomorphism.*

Proof. The uniqueness follows directly from the fact that BG has affine diagonal and the usual version of Hartogs's theorem for maps into affine schemes. By [Con14, Prop. 3.1.3] we have an exact sequence of group schemes

$$1 \to G_0 \xrightarrow{i} G \xrightarrow{q} \pi_0(G) \to 1,$$

where G_0 is reductive with connected fibers and $\pi_0(G)$ is finite étale [Alp14, Thm. 9.7.6]. By uniqueness and étale descent, it suffices to check the existence of $\widetilde{f}$ étale locally on S, and so we can assume that the neutral component G_0 is split reductive and $\pi_0(G)$ is a constant group scheme over S.

Suppose first that $G = G_0$. In this case G can be embedded as a closed subgroup $G \hookrightarrow (\mathrm{GL}_n)_S$ for some $n > 0$. By [Alp14, Th. 9.4.1] the quotient $(\mathrm{GL}_n)_S/G$ is affine, and so it follows that the morphism $BG \to B(\mathrm{GL}_n)_S$ is affine. By using Hartogs's theorem for maps into affine schemes, we are reduced to the case when $G = (\mathrm{GL}_n)_S$. In this case $f : U \to B(\mathrm{GL}_n)_S$ corresponds to a rank n vector bundle $\mathcal{E}$ on U, and we can extend to a vector bundle $\widetilde{\mathcal{E}}$ on Y by setting $\widetilde{\mathcal{E}} = (j_*\mathcal{E})^{\vee\vee}$, where $j : U \hookrightarrow Y$ is the open immersion [Sta23, Tag 0B3N] .

Now we proceed to prove the lemma for a general G. Consider the composition $g : U \to BG \to B\pi_0(G)$, corresponding to a finite étale $\pi_0(G)$-torsor $F \to U$. The morphism g admits an extension $\widetilde{g} : Y \to B\pi_0(G)$, by Zariski-Nagata purity ([Sta23, Tag 0BMA] + [Sta23, Tag 0EY7]) and using [Sta23, Tag 0BQG] to extend the $\pi_0(G)$-action. Let $p : \widetilde{F} \to Y$ denote the finite étale torsor corresponding to $\widetilde{g} : Y \to B\pi_0(G)$, and set $V = p^{-1}(U)$ to be the inverse image of U in $\widetilde{F}$. We denote by a E_V the G-bundle on V corresponding to the composition $V \to U \to BG$. By construction, the pullback $p^*(\widetilde{F})$ of the torsor is canonically trivialized. In particular, the associated $\pi_0(G)$-bundle $q_*(E_V)$ is trivialized, and so we can view E_V as a G_0-bundle on V. By the result for G_0, we can extend this to a G_0-bundle E on $\widetilde{F} \supset V$, and the associated G-bundle $i_*(\widetilde{E})$ yields an extension of E_V to $\widetilde{F}$. By étale descent, in order to descent the G-bundle $i_*(\widetilde{E})$ to Y we need to equip it with an equivariant structure for the Galois group of the cover $\Gamma = \pi_0(G)$. The set of such equivariant structures is in natural in bijection with sections of certain affine morphism $Z \to \widetilde{F}$ (cf. the last paragraph in the proof of [HLH23, Prop. 7.6]). Since the restriction E_V comes as a pullback of a G-bundle on

U, we have a section $s : V \to Z$ defined on the open $V \subset \widetilde{F}$. By Hartogs's theorem for affine morphisms, this section extends uniquely to $\widetilde{s} : \widetilde{F} \to Z$, which allows us to descend the G-bundle $i_*(\widetilde{E})$ to get our desired extension $\widetilde{f} : Y \to BG$. □

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References

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