Geometric quotients with trivial stabilizers are torsors

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This note is based on a question by user Math Display on Mathoverflow.

Let us start with the context that we will work on (the reader that needs greater generality will not have trouble modifying the arguments to work over a base scheme isntead of a field).

Context 1. *Let k be a field, and let G be an affine group scheme over k. Let X be a finite type k-scheme with a G-action, and suppose that there is a morphism $\pi : X \to Y$ to a Noetherian scheme Y such that the following are satisfied:*

(1) π is of finite type, surjective, and G-invariant.

(2) We have $\mathcal{O}_Y = (\pi_* \mathcal{O}_X)^G$.

(3) The morphism $j : G \times X \to X \times X$ given by $(g, x) \mapsto (x, g \cdot x)$ induces a surjective morphism $G \times X \to X \times_Y X$.

(4) The G-stabilizer of any geometric point of X is trivial.

The following condition will be useful.

Definition 2. *We say that the action of G on X is proper relative to Y if the induced morphism $G \times X \to X \times_Y X$ is proper.*

With this in mind, the following is not too difficult.

Proposition 3. *In Context 1, assume that the G-action is proper. Then, the morphism $\pi : X \to Y$ is a principal G-bundle.*

Proof. The condition (4) on the stabilizers means that the quotient stack X/G is actually an algebraic space; equivalently $G \times X \to X \times X$ is a monomorphism. By definition, the quotient morphism $X \to X/G$ is a principal G-bundle. There is an induced finite type morphism $f : X/G \to Y$; it suffices to show that f is an isomorphism. This can be checked Zariski locally on Y, so we may assume that Y and X are affine for concreteness. The assumption of that the action is proper relative to Y is equivalent to the morphism $f : X/G \to Y$ being separated. On the other hand, (3) implies that f induces an injection on geometric points, and so f is quasifinite in addition to separated. By [Sta24, Tag 03XX], it follows that X/G is a scheme. Notice that $H^0(\mathcal{O}_{X/G}) = H^0(\mathcal{O}_X)^G = H^0(\mathcal{O}_Y)$, where the last equality is by (2). Therefore $f_*(\mathcal{O}_{X/G}) = \mathcal{O}_Y$. By Zariski's main theorem [Sta24, Tag 02LR], it follows that f is an open immersion. The surjectivity in (1) implies that the open immersion $X/G \to Y$ is surjective, and therefore $X/G \to Y$ is an isomorphism. □

Now, I think(?) that the condition of properness of G is automatic in Context 1. I am including an argument below, which the reader may want to double-check (as it seemed a bit surprising to me).

Proposition 4. *In Context 1, the action is automatically proper. In particular, in Context 1 the morphism $\pi : X \to Y$ is always a principal G-bundle.*

Proof. By the triviality of stabilizers we have that $G \times X \to X \times X$ is a monomorphism, and it factors through a monomorphism $i : G \times X \to X \times_Y X$. We need to show that i is a closed immersion, which, by item (6) in [Sta24, Tag 04XV], is equivalent to showing that i is universally closed. The morphism $i : G \times X \to X \times_Y X$ is G-equivariant, where G acts on the first coordinate of the source by multiplication, and it acts on the second coordinate of the target by the action on X. These actions are free, and taking quotients we get a monomorphism of algebraic spaces $\widetilde{i} : X \to X \times_Y (X/G)$. Note that $G \times X \to X$ and $X \times_Y X \to X \times_Y (X/G)$ are principal G-bundles, and by working flat locally on $X \times_Y (X/G)$ one can see that the following square is Cartesian:

$$\begin{array}{ccc} G \times X & \xrightarrow{i} & X \times_Y X \\ \downarrow & & \downarrow \\ X & \xrightarrow{\widetilde{i}} & X \times_Y (X/G). \end{array}$$

Therefore it suffices to show that $\widetilde{i}$ is universally closed. Notice that, by (3), the morphism i is a surjective monomorphism. It follows that $\widetilde{i}$ is also a surjective monomorphism, in particular it is universally bijective on points. Furthermore, the morphism $\widetilde{i}$ has a section $p : X \times_Y (X/G) \to X$ given by the first projection. To check that $\widetilde{i}$ is universally closed, choose a morphism $T \to X \times_Y (X/G)$ from a scheme T and form the base-change $\widetilde{i}_T : X_T \to T$; we need to show that $\widetilde{i}_T$ is closed. But it is still the case that $\widetilde{i}_T$ is a surjective monomorphism (so it is bijective on topological points) and has a section $p_T : T \to X_T$. Using this, we see that, for given closed subset $Z \subset |X_T|$of the topological space, the image $\widetilde{i}_T(Z)$ coincides with the preimage $(p_T)^{-1}(Z)$, and so it is closed, as desired. □

References

[Sta24] The Stacks Project Authors. *Stacks Project.* `https://stacks.math.columbia.edu`, 2024.