A local criterion for smoothness

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Abstract

This is a proof of a local criterion for smoothness of a morphism between smooth schemes over a base. The proof is a small diagram chase using the cotangent complex. This was inspired by Sean Cotner's post [Cot21] in Thuses. In particular, Corollary 3 below can be used to extend the converse in the main proposition in [Cot21] to positive characteristic.

1 The relative cotangent complex

We will use cohomological indexing convention for complexes. For a morphism of rings $A \to B$, we denote by $L_{B/A}$ the cotangent complex for the morphism. More specifically, we write $L_{B/A}$ for a representative in the quasi-isomorphism class that has projective terms over B and is bounded above (e.g. the one obtained from a semi-free simplicial resolution of the A-algebra B).

We will use the following facts about the cotangent complex.

(1) **The distinguished triangle** *[Sta21, Tag 08QR]*

Any chain of morphisms of rings $A \to B \to C$ induces an exact triangle in the derived category of C-modules

$$L_{B/A} \otimes C \to L_{C/A} \to L_{C/B}$$

(2) **Morphisms of triangles** *(Using [Sta21, Tag 08QL])*

A commutative diagram of morphisms

$$\begin{array}{ccccc} A_1 & \longrightarrow & B_1 & \longrightarrow & C_1 \\ \downarrow & & \downarrow & & \downarrow \\ A_2 & \longrightarrow & B_2 & \longrightarrow & C_2 \end{array}$$

Induces a morphism of triangles of C_1-modules

$$\begin{array}{ccccc} L_{B_1/A_1} \otimes C_1 & \longrightarrow & L_{C_1/A_1} & \longrightarrow & L_{C_1/B_1} \\ \downarrow & & \downarrow & & \downarrow \\ L_{B_2/A_2} \otimes C_2 & \longrightarrow & L_{C_2/A_2} & \longrightarrow & L_{C_2/B_2} \end{array}$$

By adjunction, we get the following triangle of C_2-modules.

$$\begin{array}{ccccc} L_{B_1/A_1} \otimes C_2 & \longrightarrow & L_{C_1/A_1} \otimes C_2 & \longrightarrow & L_{C_1/B_1} \otimes C_2 \\ \downarrow & & \downarrow & & \downarrow \\ L_{B_2/A_2} \otimes C_2 & \longrightarrow & L_{C_2/A_2} & \longrightarrow & L_{C_2/B_2} \end{array}$$

(3) **Cotangent complex vs. Kahler differentials** *(Using the identification [Sta21, Tag 08R6] of the truncation with the naive cotangent complex defined in [Sta21, Tag 00S0])*

For any chain of morphisms $A \to B \to C$, we have $H^0(L_{C/A}) = \Omega^1_{C/A}$ and $H^0(L_{B/A}) = \Omega^1_{B/A}$.

Moreover $H^0(L_{B/A} \otimes C) = H^0(L_{B/A}) \otimes C = \Omega^1_{B/A} \otimes C$, and the morphism

$$\Omega^1_{B/A} \otimes C = H^0(L_{B/A} \otimes C) \to H^0(L_{C/A}) = \Omega^1_{C/A}$$

coincides with the usual morphism on Kahler differentials.

(4) **Vanishing for smooth morphisms** *([Sta21, Tag 08R5] + [Sta21, Tag 08R3] + [Sta21, Tag 08R2])*

If $k \to A$ is the localization of a smooth morphism, then $H^i(L_{A/k}) = 0$ for $i \neq 0$.

(5) **Behavior under smooth morphisms** *(By using the previous fact (4) and the triangle (1))*

Consider $k \to A \to B$, where the second morphism is a localization of a smooth morphism. Then the morphism $l_{A/k} \otimes B \to L_{B/k}$ induces an injection $H^0(L_{A/k} \otimes B) \hookrightarrow H^0(L_{B/k})$ and an isomorphism $H^i(L_{A/k} \otimes B) \xrightarrow{\sim} H^i(L_{B/k})$ for $i \neq 0$.

(6) **Cotangent complex for closed immersions** *[Sta21, Tag 08RA]*

If $A \twoheadrightarrow B$ with kernel I, then we have $H^i(L_{B/A}) = 0$ for $i > -1$ and $H^{-1}(L_{B/A}) = I/I^2$. In particular $H^{-1}(L_{B/A} \otimes C) = H^{-1}(L_{B/A}) \otimes C$ for any B-algebra C. (This last compatibility with base change is a general fact for the highest nonzero cohomology of any bounded above complex of B-flat modules).

(7) **Functoriality for closed immersions** *(By inspecting the construction of the morphism in [Sta21, Tag 08QL] (or using the identification [Sta21, Tag 08R6] of the truncation with the naive cotangent complex [Sta21, Tag 00S0]).)*

For any commutative square of rings A, B with ideals I, J as follows

$$\begin{array}{ccc} A & \longrightarrow & A/I \\ \downarrow & & \downarrow \\ B & \longrightarrow & B/J \end{array}$$

the morphism

$$I/I^2 \otimes B/J = H^{-1}(L_{(A/I)/A}) \otimes B/J \cong H^{-1}(L_{(A/I)/A}) \otimes B/J) \to H^{-1}(L_{(B/J)/B}) = J/J^2$$

is the natural one induced by the morphism of rings $A \to B$.

2 The local criterion

Proposition 1 (A local criterion for smoothness)**.** *Let k be a Noetherian ring, and let $f : X \to Y$ be a morphism of k-schemes locally of finite type over k. Let $x \in X$ and set $y = f(x)$. Suppose that the following are satisfied*

(a) X is k-smooth at x, and Y is k-smooth at y.

(b) The extension of residue fields $\kappa(y) \to \kappa(x)$ is separable.

(c) The induced morphism $\mathfrak{m}_y/\mathfrak{m}_y^2 \otimes \kappa(x) \to \mathfrak{m}_x/\mathfrak{m}_x^2$ is injective.

Then, f is smooth at x.

Proof. We will use the diagram of local rings

$$\begin{array}{ccccc} k & \longrightarrow & \mathcal{O}_{Y,y} & \longrightarrow & \kappa(y) \\ \downarrow & & \downarrow & & \downarrow \\ k & \longrightarrow & \mathcal{O}_{X,x} & \longrightarrow & \kappa(x) \end{array}$$

This yields the morphism of triangles

$$\begin{array}{ccccc} L_{\mathcal{O}_{Y,y}/k} \otimes \kappa(x) & \longrightarrow & L_{\kappa(y)/k} \otimes \kappa(x) & \longrightarrow & L_{\kappa(y)/\mathcal{O}_{Y,y}} \otimes \kappa(x) \\ \downarrow & & \downarrow & & \downarrow \\ L_{\mathcal{O}_{X,x}/k} & \longrightarrow & L_{\kappa(y)/k} \otimes \kappa(x) & \longrightarrow & L_{\kappa(x)/\mathcal{O}_{X,x}} \end{array}$$

We therefore get a morphism of long exact sequences

$$
\begin{array}{ccccccccccc}
H^{-1}(L_{\mathcal{O}_{Y,y}/k} \otimes \kappa(x)) & \to & H^{-1}(L_{\kappa(y)/k} \otimes \kappa(x)) & \to & H^{-1}(L_{\kappa(y)/\mathcal{O}_{Y,y}} \otimes \kappa(x)) & \to & H^0(L_{\mathcal{O}_{Y,y}/k} \otimes \kappa(x)) & \to & H^0(L_{\kappa(y)/k} \otimes \kappa(x)) & \to & H^0(L_{\kappa(y)/\mathcal{O}_{Y,y}} \otimes \kappa(x)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^{-1}(L_{\mathcal{O}_{X,x}/k} \otimes \kappa(x)) & \longrightarrow & H^{-1}(L_{\kappa(x)/k}) & \longrightarrow & H^{-1}(L_{\kappa(x)/\mathcal{O}_{X,x}}) & \longrightarrow & H^0(L_{\mathcal{O}_{X,x}/k} \otimes \kappa(x)) & \longrightarrow & H^0(L_{\kappa(x)/k}) & \longrightarrow & H^0(L_{\kappa(x)/\mathcal{O}_{X,x}})
\end{array}
$$

Now we can use that both $k \to \mathcal{O}_{X,x}$ and $k \to \mathcal{O}_{Y,y}$ are localizations of smooth morphisms. Using the facts about the cotangent complex dicussed in the previous section, we get

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & H^{-1}(L_{\kappa(y)/k} \otimes \kappa(x)) & \longrightarrow & \mathfrak{m}_y/\mathfrak{m}_y^2 \otimes \kappa(x) & \longrightarrow & \Omega^1_{\mathcal{O}_{Y,y}/k} \otimes \kappa(x) & \longrightarrow & H^0(L_{\kappa(y)/k} \otimes \kappa(x)) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \hookdownarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & H^{-1}(L_{\kappa(x)/k}) & \longrightarrow & \mathfrak{m}_x/\mathfrak{m}_x^2 & \longrightarrow & \Omega^1_{\mathcal{O}_{X,x}/k} \otimes \kappa(x) & \longrightarrow & H^0(L_{\kappa(x)/k}) & \longrightarrow & 0
\end{array}
$$

Since the extesion $\kappa(y) \to \kappa(x)$ is separable and finitely generated, it is a localization of a smooth morphism. We conclude that $L_{\kappa(y)/k} \otimes \kappa(x) \to L_{\kappa(x)/k}$ induces an injection $H^0(L_{k(y)/k} \otimes k(x)) \hookrightarrow H^0(L_{k(x)/k})$ and an isomorphism $H^{-1}(L_{k(y)/k} \otimes k(x)) \xrightarrow{\sim} H^{-1}(L_{k(x)/k})$. Using the fact that Kahler differentials commute with localization, we can rewrite the diagram above as

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & H^{-1}(L_{\kappa(y)/k} \otimes \kappa(x)) & \longrightarrow & \mathfrak{m}_y/\mathfrak{m}_y^2 \otimes \kappa(x) & \longrightarrow & f^*\Omega^1_{Y/k} \otimes \kappa(x) & \longrightarrow & H^0(L_{\kappa(y)/k} \otimes \kappa(x)) & \longrightarrow & 0 \\
\downarrow & & \downarrow\wr & & \hookdownarrow & & \downarrow & & \hookdownarrow & & \downarrow \\
0 & \longrightarrow & H^{-1}(L_{\kappa(x)/k}) & \longrightarrow & \mathfrak{m}_x/\mathfrak{m}_x^2 & \longrightarrow & \Omega^1_{X/k} \otimes \kappa(x) & \longrightarrow & H^0(L_{\kappa(x)/k}) & \longrightarrow & 0
\end{array}
$$

A diagram chase now shows that the natural morphism $f^*\Omega^1_{Y/k} \otimes \kappa(x) \to \Omega^1_{X/k} \otimes \kappa(x)$ is injective. By [BLR90, §2.2, Prop 8], it follows that f is smooth at x. □

Remark 2. *Thanks to Sean Cotner for pointing out that the separability of the extension in (b) is enough (originally I also had the assumption that the extension is algebraic).*

As a corollary, we get the following special case.

Corollary 3. *Let k be a field and let $f : X \to Y$ be a morphism of smooth k-schemes. Let $x \in X$ and set $y = f(x)$. Suppose that the residue fields $\kappa(x)$ and $\kappa(y)$ coincide, and the induced morphism on cotangent spaces $\mathfrak{m}_y/\mathfrak{m}_y^2 \to \mathfrak{m}_x/\mathfrak{m}_x^2$ is injective. Then, f is smooth at x.*

References

[BLR90] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud. *Néron models*, volume 21 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1990.

[Cot21] Sean Cotner. A curiosity: "supersmooth" varieties. https://thuses.com/algebraic-geometry/a-curiosity-supersmooth-varieties/, 2021. Thuses post.

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