Cartier descent as fppf descent

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Cartier descent [Kat70, Theorem 5.1] is the following statement.

Theorem 1. *Given any quasi-coherent sheaf V' on C'/S, there is a canonical connection ∇^{can} on the pull back $F^*_{C_S/S}V'$ with vanishing p-curvature such that the horizontal part $(F^*_{C_S/S}V')^{\nabla^{can}}$ descends to V'. Conversely, any quasi-coherent sheaf with a flat connection (V, ∇) on C/S whose p-curvature vanishes is isomorphic to $(F^*_{C_S/S}V', \nabla^{can})$ for some V' on C'/S.*

Let us explain how Cartier descent is indeed a consequence of fppf descent. This is roughly pointed out in [Gro59, p.321] and [Her21, Example A.2], but not stated explicitly.

Given any smooth S-scheme X, the relative Frobenius morphism $F_{X/S} : X \to X'$ is fppf and the pushforward $F_{X/S,*}\mathcal{O}_X$ is a locally free sheaf over X'. Let E be a quasi-coherent sheaf on X. Let $\pi : X \times_{X'} X \to X'$ be the natural projection. There are natural $\pi_*\mathcal{O}_{X\times_{X'}X}$-module structures on the sheaves $End_{\mathcal{O}_{X'}}(F_{X/S,*}\mathcal{O}_X)$ and $End_{\mathcal{O}_{X'}}(F_{X/S,*}E)$ given by multiplications on the source and target. Since π is affine we have two corresponding $\mathcal{O}_{X\times_{X'}X}$-algebras denoted by $\widetilde{End}_{\mathcal{O}_{X'}}(\mathcal{O}_X)$ and $\widetilde{End}_{\mathcal{O}_{X'}}(E)$. Using the fact that $F_{X/S,*}\mathcal{O}_X$ is locally free, and using the hom-tensor adjunction repeatedly, one can deduce that a descent datum for E relative to $F_{X/S}$ is equivalent to a homomorphism $\widetilde{End}_{\mathcal{O}_{X'}}(\mathcal{O}_X) \to \widetilde{End}_{\mathcal{O}_{X'}}(E)$ of $\mathcal{O}_{X\times_{X'}X}$-algebras.

Let $D_{X/S}$ be the sheaf of crystalline differential operators on X/S. It is the universal enveloping algebra for the relative tangent sheaf $T_{X/S}$. There is a natural morphism $i : D_{X/S} \to End_{\mathcal{O}_S}(\mathcal{O}_X)$ of $\mathcal{O}_X$-algebras which is not injective since ∂^p and $\partial^{[p]}$ have the same image. Since $\mathcal{O}_{X'}$ sits naturally in the center of $F_{X/S,*}D_{X/S}$, we have a natural morphism $F_{X/S,*}i : F_{X/S,*}D_{X/S} \to End_{\mathcal{O}_{X'}}(F_{X/S,*}\mathcal{O}_X)$ on X'. Using the explicit description of the latter ring in [Her21, Example A.2], or the Azumaya algebra property of $F_{X/S,*}D_{X/S}$ over $\mathcal{O}_{T^*X'}$ as in [BMRR08], we see that $F_{X/S,*}i$ coincides with the quotient morphism by the relation generated by $\partial^p - \partial^{[p]}$. Furthermore, the left and right multiplications of $\mathcal{O}_X$ on $D_{X/S}$ gives a natural $\mathcal{O}_{X\times_{X'}X}$-algebra structure on $F_{X/S,*}$, and $F_{X/S,*}i$ respects the algebra structure.

Combining the two paragraphs above, we see that a descent data for E relative to $F_{X/S}$ is exactly a morphism of $\mathcal{O}_{X\times_{X'}X}$-algebras $D_{X/S} \to End_{\mathcal{O}_S}(E)$ that maps all sections of the form $\partial^p - \partial^{[p]}$ to 0, i.e., a flat connection on E with trivial p-curvature. We have thus recovered Cartier descent from fppf descent.

References

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