Canonical factorizations of finite morphisms of reduced stacks

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Abstract

The main result of this note is Proposition 5. It states that, under mild hypotheses, a finite schematic morphism of reduced stacks factors canonically as the composition of a finite radicial morphism composed with a finite generically étale morphism.

Definition 1. *Let $f\colon X \to Z$ be a finite and surjective morphism of reduced schemes of finite type over a field k. A blowup of Z in X is defined to be a factorization $X \xrightarrow{f'} Z' \xrightarrow{\beta} Z$ over k with Z' reduced, and β finite and birational as in [Sta22, Tag 01RN] (i.e. β induces a bijection between the sets of generic points and an isomorphism at the level of local rings of corresponding generic points). We say that Z doesn't have blowups in X, if for any blowup of Z in X, then the morphism β is the identity.*

In this paragraph, we characterize the absence of blowups when X and Z are integral. Let $j_Z\colon \eta_Z \to Z$ be the inclusion of the generic point of Z. The skyscraper sheaf $\mathcal{O}_{\eta_Z}$ with stalk the field $K(Z)$ has pushforward $j_{Z*}\mathcal{O}_{\eta_Z}$, a quasicoherent sheaf of $\mathcal{O}_Z$-algebras. We have two natural inclusions:

(1) $\mathcal{O}_Z \subset j_{Z*}\mathcal{O}_{\eta_Z} \subset f_* j_{X*}\mathcal{O}_{\eta_X}$, the latter a skyscraper sheaf with stalk $K(X)$ at η_Z;

(2) $\mathcal{O}_Z \subset f_*\mathcal{O}_X \subseteq f_* j_{X*}\mathcal{O}_{\eta_X}$.

The intersection $\mathcal{A} := j_{Z*}\mathcal{O}_{\eta_Z} \cap f_*\mathcal{O}_X$ yields a quasicoherent sheaf of $\mathcal{O}_Z$-algebras, which is finitely generated as an $\mathcal{O}_Z$-module, since it sits inside the coherent $\mathcal{O}_Z$-module $f_*\mathcal{O}_X$ and $\mathcal{O}_Z$ is coherent. Note that by construction, $\mathcal{A}$ is the maximal coherent $\mathcal{O}_Z$-algebra sitting inside $f_*\mathcal{O}_X$ that agrees with $\mathcal{O}_Z$ at the generic point η_Z. Set $\overline{Z} := \underline{\mathrm{Spec}}_Z(\mathcal{A})$. We have a factorization $f\colon X \xrightarrow{\overline{f}} \overline{Z} \xrightarrow{\overline{\beta}} Z$ where $\overline{\beta}$ is birational, and the maximality of $\mathcal{A}$ implies that any blowup of Z in X is dominated by $\overline{Z}$. In particular, $\overline{Z}$ doesn't have any blowups in X. It follows that Z doesn't have any blowups in X if and only if $\overline{Z} = Z$ if and only if $\mathcal{A} := j_{Z*}\mathcal{O}_{\eta_Z} \cap f_*\mathcal{O}_X = \mathcal{O}_Z$. Note that this can be checked Zariski locally on Z, so that the property of not having blowups in X is Zariski local on the target Z.

We can also consider the more general case when $f\colon X \to Z$ is a finite surjective morphism of (possibly reducible) reduced schemes of finite type over k, under the assumption that every irreducible component $X_i \subset X$ dominates an irreducible component of Z. Let I_Z be the set of generic points of the irreducible components of Z. For any $z \in I_Z$, we have the residue field $\mathcal{O}_z = k(z)$ and the evident monomorphism

$j_Z\colon \eta_Z \coloneqq \coprod_{z\in I_Z} z \to Z$ inducing the natural inclusion $\mathcal{O}_Z \subseteq j_{Z*}\mathcal{O}_{\eta_Z} = \oplus_{z\in I_Z}\mathcal{O}_z$. For X and also for other forthcoming schemes, we define similarly I_X. The morphism f induces a surjection $I_f\colon I_X \to I_Z$ and an inclusion $j_{Z*}\mathcal{O}_Z \subseteq f_* j_{X*}\mathcal{O}_{\eta_X}$. In complete analogy with the case when X and Z are integral, we have that Z doesn't have any blowups in X if and only if $\mathcal{O}_Z = j_{Z*}\mathcal{O}_{\eta_Z} \cap f_*\mathcal{O}_X \subseteq f_* j_{X*}\mathcal{O}_{\eta_X}$. In particular, for such morphisms, having no blowups can be checked Zariski locally on the target Z.

In the next lemma, we show that the property of having no blowups is also smooth local on the target.

Lemma 2. *Let $f\colon X \to Z$ be a finite surjective morphism of reduced schemes of finite type over k. Suppose that every irreducible component of X dominates an irreducible component of Z. Let $U \to Z$ be a smooth surjective morphism from a scheme U of finite type over k, thus inducing a base-change morphism $f_U\colon X_U \to U$ of reduced schemes. Then Z doesn't have any blowups in X if and only if U doesn't have any blowups in X_U.*

Proof. Since $U \to Z$ and $X_U \to X$ are smooth, both U and X_U are reduced. Moreover, the image of any irreducible component of X_U is an irreducible component of U, for this image contains an open subset of U.

The "only if" direction can be seen directly: if Z admits a nontrivial blowup $X \to Z' \to Z$ in X, then the base-change $X_U \to (Z')_U \to U$ is a nontrivial blowup of U in X_U.

Next, we prove the "if" direction. Suppose that Z doesn't have any blowups in X. We want to show that U doesn't have any blowups on X_U. We can check this Zariski locally on U and Z, so we can assume without loss of generality that $U \to Z$ induces a bijection $I_U = I_Z$ on the respective sets of generic points. Let $j_Z\colon \eta_Z = \sqcup_{z\in I_Z} z \to Z$ be the inclusion of the generic points of irreducible components of Z, and similarly let $j_U\colon \eta_U \to U$ and $j_{X_U}\colon \eta_{X_U} \to X_U$. In order to show that U doesn't have any blowups in X_U, we need to show that $(j_U)_*\mathcal{O}_{\eta_U} \cap (f_U)_*\mathcal{O}_{X_U} = \mathcal{O}_U$.

By assumption, we know that $(j_Z)_*\mathcal{O}_{\eta_Z} \cap f_*\mathcal{O}_X = \mathcal{O}_Z$. Let $(j_Z)_U\colon (\eta_Z)_U \to U$ denote the base-change of j_Z by U. Since $U \to Z$ is flat, by flat base-change and exactness of pullback we can conclude that

$$((j_Z)_U)_*\mathcal{O}_{(\eta_Z)_U} \cap (f_U)_*\mathcal{O}_{X_U} = \mathcal{O}_U \tag{1}$$

For each $z \in I_Z$, the scheme z_U is normal, since it is smooth over the normal point z. Moreover, the monomorphism $z_U \to U$ is affine and locally given as a localization, because the same holds for $z \to Z$. Since $z_U \to U$ factors through an irreducible component of $U_i \subset U$ (because of the bijection $I_U = I_Z$), we conclude that z_U is integral, with fraction field $k(z_U) = \mathcal{O}_u$, where u is the corresponding generic point in U. The inclusion $j_U\colon \eta_U = \sqcup_{u\in I_U} u \to U$ factors as $\sqcup_{u\in I_U} u \xrightarrow{h} \sqcup_{z\in I_Z} z_U \to U$, exhibiting each u as the generic point of the corresponding z_U. In particular $\mathcal{O}_{\eta_U}$ is the total ring of fractions of the reduced scheme $(\eta_Z)_U$.

Since $(\eta_Z)_U$ is normal and the base-change $f_{(\eta_Z)_U}\colon X_{(\eta_Z)_U} \to z_U$ is finite, it follows that $h_*\mathcal{O}_{\eta_U} \cap (f_{(\eta_Z)_U})_*\mathcal{O}_{X_{(\eta_Z)_U}} = \mathcal{O}_{(\eta_Z)_U}$. By pushing forward via $(j_Z)_U$, it follows that

$$(j_U)_*\mathcal{O}_{\eta_U} \cap ((j_Z)_U \circ f_{(\eta_Z)_U})_*\mathcal{O}_{X_{(\eta_Z)_U}} = ((j_Z)_U)_*\mathcal{O}_{(\eta_Z)_U}$$

Notice that there is an inclusion $(f_U)_*\mathcal{O}_{X_U} \subset ((j_Z)_U \circ f_{(\eta_Z)_U})_*\mathcal{O}_{X_{(\eta_Z)_U}}$, obtained by applying $(f_U)_*$ to the inclusions of $\mathcal{O}_{X_U}$-algebras induced by the dominant morphism

$X_{(\eta_Z)_U} \to X_U$. Intersecting the previous equality with $(f_U)_*\mathcal{O}_{X_U}$ we obtain

$$(j_U)_*\mathcal{O}_{\eta_U} \cap (f_U)_*\mathcal{O}_{X_U} = ((j_Z)_U)_*\mathcal{O}_{(\eta_Z)_U} \cap (f_U)_*\mathcal{O}_{X_U}$$

Using the equality (1) above we get $(j_U)_*\mathcal{O}_{\eta_U} \cap (f_U)_*\mathcal{O}_{X_U} = \mathcal{O}_U$, as desired. □

We will also need to following lemma, which is a generalization of the radicial claim in the proof of [dCHL18, Lemma 4.4.2].

Lemma 3. *Let $f\colon X \to Z$ be a finite surjective morphism of reduced schemes of finite type over k that induces a bijection of irreducible components. Suppose that f is generically radicial (i.e. each restriction to irreducible components $f\colon X_i \to Z_i$ induces a purely inseparable extension of fraction fields). If Z doesn't have any blowups in X, then f is radicial.*

Proof. Suppose first that the characteristic of k is 0, then f being generically radicial implies that $X \to Z$ is a finite birational morphism. Since Z doesn't have any blowups in X, we must have $X = Z$, so f is radicial.

Suppose otherwise that the characteristic of k is $p > 0$. We claim that there exists some positive integer n such that $(\pi_*\mathcal{O}_X)^{p^n} \subset \mathcal{O}_Z$. Indeed, the assumption that f is generically radicial implies that there is some n such for all irreducible components $X_i \to Z_i$ we have $(k(X_i))^{p^n} \subset k(Z_i)$. This in particular implies that $(\pi_*\mathcal{O}_X)^{p^n} \subset (j_Z)_* \prod_i k(Z_i) = (j_Z)_*\mathcal{O}_{\eta_Z}$, where $j_Z\colon \eta_Z = \sqcup_{z\in I_Z} z \to Z$ is the inclusion of the generic points of Z. The assumption on Z having no blowups in X implies that $(j_Z)_*\mathcal{O}_{\eta_Z} \cap \pi_*\mathcal{O}_X$, so we conclude that

$$(\pi_*\mathcal{O}_X)^{p^n} \subset (j_Z) * \mathcal{O}_{\eta_Z} \cap \pi_*\mathcal{O}_X = \mathcal{O}_Z$$

Once we know that $(\pi_*\mathcal{O}_X)^{p^n} \subset \mathcal{O}_Z$, then we can see that f is radicial by working over affine opens of Z and applying the same argument as in the claim in the proof of [dCHL18, Lemma 4.4.2]. □

If $\pi\colon X \to Y$ is a finite and surjective morphism of integral finite type schemes over k, then it induces an inclusion $k(Y) \subset k(X)$ of fraction fields. For for any intermediate field $k(Y) \subset F \subset k(X)$, there is a unique factorization into finite k-morphisms of integral k-schemes $X \overset{f_F}{\to} Z_F \overset{g_F}{\to} Y$ such that:

(a) The fraction field of Z_F is identified with the intermediate field F.

(b) Z_F doesn't have any blowups in X.

Indeed, we can take $Z_F = \underline{\mathrm{Spec}}_Y((j_Y)_*F \cap \pi_*\mathcal{O}_X)$, where $j_Y\colon \eta_Y \to Y$ is the inclusion of the generic point of Y. Furthermore, it follows from construction that all other factorizations $X \to Z' \to Y$ satisfying property (a) above are dominated by Z_F, in other words there is a factorization $X \to Z_F \to Z' \to Y$. So Z_F is initial among all intermediate schems $X \to Z' \to Y$ satisfying (a).

More generally, we can relax the assumptions for the previous discussion and only assume that X and Y reduced, and that each irreducible component of X dominates an irreducible component of Y. Then, for each such component $X_i \subset X$ with image $Y_i \subset Y$, we get inclusions of fraction fields $k(Y_i) \subset K(X_i)$. If we choose some intermediate field $K(Y_i) \subset F_i \subset K(X_i)$ for each i, then we can set $Z_{(F_i)}$ to be the reduced scheme

$\underline{\mathrm{Spec}}((j_Y)_* \prod_i F_i \cap \pi_* \mathcal{O}_X)$. Here $j_Y : \eta_Y = \sqcup_{y \in I_Y} y \to Y$ is the inclusion of the generic points of Y, and we view the product $\prod_{i \in I_Z} F_i \subset \prod_{i \in I_X} k(X_i) = \mathcal{O}_{\eta_X}$ as a sheaf on η_Y via pushforward under $\eta_X \to \eta_Y$. We then obtain the unique factorization

$$X \xrightarrow{f_{(F_i)}} Z_{(F_i)} \xrightarrow{g_{(F_i)}} Y \tag{2}$$

of finite surjective morphisms such that

(a) $f_{(F_i)}$ induces a bijection of irreducible components.

(b) For each connected component X_i with corresponding component $C_i \subset Z_{(F_i)}$, the inclusion of fraction fields $k(Y_i) \subset k(C_i) \subset k(X_i)$ induces an identificaton $k(C_i) = F_i$.

(c) $Z_{(F_i)}$ does not have any blowups in X.

In addition, it follows from construction that $Z_{(F_i)}$ is initial among all intermediate schemes $X \to Z' \to Y$ satisfing properties (a) and (b) above.

Proposition 4 (Canonical factorization of finite morphisms of reduced schemes)**.** *Let $\pi : X \to Y$ be an finite surjective morphism of reduced schemes of finite type over k. Suppose that every irreducible component of X dominates an irreducible component of Y. There is a factorization $X \xrightarrow{f} Z \xrightarrow{g} Y$ such that f is radicial, g is finite and generically étale, and Z is an reduced that has no blowups in X. Furthermore, Z dominates any other factorization $X \xrightarrow{f'} Z' \xrightarrow{g'} Y$ with Z' integral, f' radicial and g' finite and generically étale (in other words for any such Z' there is a factorization $X \to Z \to Z' \to Y$).*

Proof. Any factorization $X \xrightarrow{f} Z \xrightarrow{g} Y$ with Z reduced and f radicial induces a bijection between irreducible components of X and irreducible components of Z. For any irreducible component $X_i \subset X$ with images $Z_i \subset Z$ and $Y_i \subset Y$, we would get a chain of inclusions of fraction fields $k(Y_i) \subset k(Z_i) \subset k(X_i)$. If in addition we assume that g is generically étale, then $k(Z_i)$ has to be the separable closure F_i of $k(Y_i)$ in $k(X_i)$. Therefore, if moreover Z has no blowups, then we must have $Z = Z_{(F_i)}$ as in (2) above. In particular the uniqueness of the desired factorization follows. This argument also shows that all factorizations $X \xrightarrow{f'} Z' \xrightarrow{g'} Y$ with f' radicial and g' finite and generically étale are dominated by $Z_{(F_i)}$.

In order to prove existence, we can take $Z := Z_{(F_i)}$ and we are only left to show that the induced morphism $f : X \to Z_{(F_i)}$ is radicial. This follows directly from Lemma 3, since f induces a bijection of connected components and is generically radicial by construction. □

Using the compatibility with smooth base-change (Lemma 2), we can now perform a formal standard argument to generalize Proposition 4 to the setting of stacks.

Proposition 5 (Canonical factorization of finite morphisms of reduced stacks)**.** *Let $\pi : \mathfrak{X} \to \mathfrak{Y}$ be a finite schematic surjective morphism of reduced stacks of finite type over k. Suppose that every irreducible component of $\mathfrak{X}$ dominates an irreducible component of $\mathfrak{Y}$. Then there is a unique initial factorization $\mathfrak{X} \xrightarrow{f} \mathfrak{Z} \xrightarrow{g} \mathfrak{Y}$ with f is schematic and radicial, g finite schematic generically étale, and $\mathfrak{Z}$ a reduced stack. (Here initial means that for any other $\mathfrak{X} \xrightarrow{f'} \mathfrak{Z}' \xrightarrow{g'} \mathfrak{Y}$ with f' finite schematic radicial, g' finite schematic generically étale, and $\mathfrak{Z}'$ reduced, there is a factorization $\mathfrak{X} \to \mathfrak{Z} \to \mathfrak{Z}' \to \mathfrak{Y}$).*

Proof. In view of the surjectivity of $\mathfrak{X} \to \mathfrak{Z}$, if such morphism exists, then we must have $g_*\mathcal{O}_{\mathfrak{Z}} \subset \pi_*\mathcal{O}_{\mathfrak{X}}$ and $\mathfrak{Z} = \underline{\mathrm{Spec}}_{\mathfrak{Y}}(g_*\mathcal{O}_{\mathfrak{Z}})$. We prove the existence of a $\mathcal{O}_{\mathfrak{Y}}$-subalgebra $\mathcal{B} \subset \pi_*\mathcal{O}_{\mathfrak{X}}$ such that the corresponding stack $\mathfrak{Z} = \underline{\mathrm{Spec}}_{\mathfrak{Y}}(\mathcal{B})$ satisfies the required properties.

Let $h\colon U \to \mathfrak{Y}$ be a smooth atlas of $\mathfrak{Y}$ with U a reduced scheme of finite type over k. Consider the diagram of Cartesian squares:

$$\begin{array}{ccccc} V \times_{\mathfrak{X}} V & \overset{p_1}{\underset{p_2}{\rightrightarrows}} & V & \xrightarrow{p} & \mathfrak{X} \\ \downarrow{\scriptstyle \pi''} & & \downarrow{\scriptstyle \pi_U} & & \downarrow{\scriptstyle \pi} \\ U \times_{\mathfrak{Y}} U & \overset{h_1}{\underset{h_2}{\rightrightarrows}} & U & \xrightarrow{h} & \mathfrak{Y}. \end{array}$$

Here, $\pi_U\colon V \to U$ is a finite surjective morphism of reduced schemes. For each irreducible component $V_i \subset V$ the image $\pi(V_i)$ contains an open subset of U, and so it is an irreducible component of U. Hence the morphism π_U satisfies the hypotheses of Proposition 4. Therefore there is an analogous subalgebra $\mathcal{C} \subset (\pi_U)_*\mathcal{O}_V$ for the morphism of schemes $\pi_U\colon V \to U$, so that the factorization $V \to \underline{\mathrm{Spec}}_U(\mathcal{C}) \to U$ satisfies the required properties. By flat base-change and the compatibility of having no blowups with smooth base-change (Lemma 2), the pullbacks $h_1^*\mathcal{C}$ and $h_2^*\mathcal{C}$ are two subalgebras of $\pi''_*\mathcal{O}_{V\times_{\mathfrak{X}}V}$ inducing factorizations of the morphism of reduced schemes $\pi''\colon V \times_{\mathfrak{X}} V \to U \times_{\mathfrak{Y}} U$ satisfying the same required properties. By the uniqueness in Proposition 4, $h_1^*\mathcal{C} = h_2^*\mathcal{C}$ as subalgebras of $\pi''_*\mathcal{O}_{V\times_{\mathfrak{X}}V}$. By smooth descent, this means that $\mathcal{C} = h^*\mathcal{B}$ for a $\mathcal{O}_{\mathfrak{Y}}$-subalgebra $\mathcal{B} \subset \pi_*\mathcal{O}_{\mathfrak{X}}$. Let $\mathfrak{X} \xrightarrow{f} \underline{\mathrm{Spec}}_{\mathfrak{Y}}(\mathcal{B}) \xrightarrow{g} \mathfrak{Y}$ be the corresponding factorization. Since being radicial and generically étale are all smooth local properties on the target of the morphism, f is finite schematic and radicial, and g is finite schematic and generically étale, by the analogous properties for the base-change $V \to \underline{\mathrm{Spec}}_U(\mathcal{C}) \to U$.

We are only left to show the last statement that $\mathfrak{X} \xrightarrow{f} \mathfrak{Z} \xrightarrow{g} \mathfrak{Y}$ is initial among all other factorizations $\mathfrak{X} \xrightarrow{f'} \mathfrak{Z}' \xrightarrow{g'} \mathfrak{Y}$ with $\mathfrak{Z}'$ reduced, f' finite schematic radicial and g' finite schematic generically étale. Any other such factorization is given by $\underline{\mathrm{Spec}}_Y(\mathcal{D})$ for some $\mathcal{O}_{\mathfrak{Y}}$-subalgebra $\mathcal{D} \subset \pi_*\mathcal{O}_{\mathfrak{X}}$. We need to show that $\mathcal{D} \subset \mathcal{B}$, where $\mathcal{B}$ is the subalgebra defined in the proof above. This can be checked by pulling back to the atlas $h\colon U \to \mathfrak{Y}$. The pullbacks $h^*\mathcal{D}$ is a $\mathcal{O}_U$-subalgebra of $(\pi_U)_*\mathcal{O}_V$ inducing the factorization $V \xrightarrow{f'_U} \underline{\mathrm{Spec}}_U(h^*\mathcal{D}) \xrightarrow{g'_U} U$ with $\underline{\mathrm{Spec}}_U(h^*\mathcal{D})$ reduced, f'_U radicial and g'_U finite and generically étale. By the universal property in Proposition 4, the factorization $V \to \underline{\mathrm{Spec}}_U(\mathcal{C}) \to U$ dominates $\underline{\mathrm{Spec}}_U(h^*\mathcal{D})$. It follows that $\mathcal{D}$ is contained in $\mathcal{C} = h^*\mathcal{B}$. By smooth descent this implies that $\mathcal{D} \subset \mathcal{B}$, as desired. □

References

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