A pathological faithfully flat morphism

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In this note I would like to record my favorite pathological example of a faithfully flat morphism. This arose when discussing with Andres Ibanez Nunez about descent. It shows why we can't expect certain properties to be truly flat-local, but instead we need to impose some local quasicompactness (as in the class of fpqc morphisms).

Construction 0.1. *Let k be an algebraically closed field, and let C be a smooth separated connected curve of finite type over k. We denote by η the generic point of C. For every closed point $p \in C(k)$, we set $C_p = \mathrm{Spec}(\mathcal{O}_{C,p})$ to be the spectrum of the local ring $\mathcal{O}_{C,p}$ at p. Every C_p contains as an open subscheme the generic point η. We denote by $\widetilde{C}$ the union of all C_p glued at the generic point η. This is a scheme, since we are just gluing in the Zariski topology. There is a morphism $\widetilde{C} \to C$.*

We note that each $C_p \subset \widetilde{C}$ is an affine open subscheme of $\widetilde{C}$. Any open subscheme $U \subset \widetilde{C}$ is of the form

$$U = U_\Sigma := \bigcup_\Sigma C_p$$

where $\Sigma \subset C(k)$ is a subset of closed points of C. The subscheme U_Σ is quasicompact if and only if Σ is a finite set of closed points.

Proposition 0.2. *The morphism $\widetilde{C} \to C$ is flat and surjective.*

Proof. Surjectivity is clear. For flatness, it is sufficient to check on the open cover $\widetilde{C} = \bigcup_{p \in C(k)} C_p$. Each morphism $C_p \to C$ is flat, as it is locally given by localizing at the prime ideal corresponding to p. □

It follows readily from the description of the topology of $\widetilde{C}$ that the morphism $\widetilde{C} \to C$ is not fpqc.

Proposition 0.3. *$\widetilde{C} \to C$ is a flat monomorphism.*

Proof. It suffices to show that for all k-algebras A, the induced morphism on A-points $\widetilde{C}(A) \to C(A)$ is an isomorphism. Suppose that $x, y \in \widetilde{C}(A)$ are two distinct A-points of $\widetilde{C}$. We want to show that their images in $C(A)$ are distinct. Since $\mathrm{Spec}(A)$ is quasicompact, the two correponding morphisms $x, y : \mathrm{Spec}(A) \to \widetilde{C}$ factor through a quasicompact open subset $U_\Sigma \subset \widetilde{C}$ with $\Sigma \subset C(k)$ finite. Therefore, it is sufficient to show that $U_\Sigma \to C$ is a monomorphism. This is clear, as it is locally given by a localization with multiplicative set the complement of the union of the finitely many primes corresponding to the closed points $\Sigma \subset C(k)$. □

This example shows that the property of being an isomorphism cannot be checked after base-change to a faithfully flat morphism. Indeed, consider the morphism $f : \widetilde{C} \to C$. This is plainly not an isomorphism, as $\widetilde{C}$ is not quasicompact. However, when we take the base-change with the same morphism $\widetilde{C} \to C$ we get the following Cartesian diagram

$$\begin{array}{ccc} \widetilde{C} & \xrightarrow{id} & \widetilde{C} \\ \Big\downarrow{\scriptstyle id} & & \Big\downarrow{\scriptstyle f} \\ \widetilde{C} & \xrightarrow{f} & C \end{array}$$

The fact that this diagram is Cartesian follows because f is a monomorphism. Even though the bottom horizontal morphism f is not an isomorphism, when we base-change via the faithfully flat morphism $\widetilde{C} \to C$ we get an isomorphism in the top horizontal arrow.