COLUMBIA UNIVERSITY

Math V1102 Sec 6
Calculus II
Fall 2015

Midterm I
10.08.2015

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Name and UNI: ______________________________________________________

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Instructions:

• There are 7 questions on this exam.

• Please write your NAME and UNI on top of EVERY page.

• Unless otherwise is explicitly stated SHOW YOUR WORK in every question.

• Please write neatly, and put your final answer in a box.

• No calculators, cell phones, books, notebooks, notes or cheat sheets are allowed.

• Useful identities:

\[
\sin^2(\theta) + \cos^2(\theta) = 1, \quad \tan^2(\theta) + 1 = \sec^2(\theta)
\]

\[
\sin(2\theta) = 2\sin(\theta)\cos(\theta), \quad \cos(2\theta) = 2\cos^2(\theta) - 1 = 1 - 2\sin^2(\theta)
\]

• The roots of the equation \( ax^2 + bx + c = 0 \) are given by

\[
x_1, x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]
1. (5 points) Determine if the following statements are true (T) or false (F). You **DO NOT** have to justify your answer for this question.

- If \( \int_1^\infty (f(x))^2 \, dx \) is convergent then \( \int_1^\infty f(x) \, dx \) is also convergent.
- If \( \int_1^\infty f(x) \, dx \) and \( \int_1^\infty g(x) \, dx \) are both divergent then \( \int_1^\infty \frac{f(x)}{g(x)} \, dx \) is also divergent.
- If \( f(x) > 0 \) and \( \int_1^\infty f(x) \, dx \) is convergent then \( \int_1^\infty \frac{f(x)}{x} \, dx \) is also convergent.
- If \( f(x) < g(x) \) and \( \int_1^\infty f(x) \, dx \) diverges then \( \int_1^\infty g(x) \, dx \) diverges.
- \( \int_0^1 x^{-\sqrt{2}} \, dx \) is divergent.

**Solution:**

(a) False. Take, for example, \( f(x) = 1/x \).

(b) False. Take, for example, \( f(x) = 1, g(x) = x^2 \).

(c) True. Comparison theorem.

(d) False. Take, for example, \( f(x) = -1, g(x) = 0 \).

(e) True. p-test.
2. (5 points) 
\[ \int x^2 \cos(x) \, dx. \]

**Solution:**

IBP: \( u = x^2, \ v' = \cos(x) \Rightarrow u' = 2x, v = \sin(x) \). Then,

\[ \int x^2 \cos(x) \, dx = x^2 \sin(x) - 2 \int x \sin(x) \, dx. \]

IBP on the last integral: \( u = x, v' = \sin(x) \Rightarrow u' = 1, v = -\cos(x) \). Then,

\[ \int x \sin(x) \, dx = -x \cos(x) + \int \cos(x) \, dx = -x \cos(x) + \sin(x) + C. \]

Therefore,

\[ \int x^2 \sin(x) \, dx = x^2 \sin(x) + 2x \cos(x) - 2 \sin(x) + C. \]
3. (10 points)

\[ \int \tan^3(x) \, dx. \]

**Solution:**

\[
\int \tan^3(x) \, dx = \int (\sec^2(x) - 1) \tan(x) \, dx \\
= \int u \, du - \int \tan(x) \, dx \\
= \frac{u^2}{2} - \ln |\sec(x)| + C \\
= \frac{\tan^2(x)}{2} - \ln |\sec(x)| + C.
\]
4. (10 points) Given that

\[ \int \sec^3(x) dx = \frac{1}{2} [\sec(x) \tan(x) + \ln(\sec(x) + \tan(x))] + C, \]

integrate

\[ \int \sec^5(x) dx. \]

**Solution:** Let \( I = \int \sec^5(x) dx \). We integrate by parts \( u = \sec^3, v' = \sec^2 \). Then,

\[
I = \tan(x) \sec^3(x) - 3 \int \sec^3(x) \tan^2(x) dx \\
= \tan(x) \sec^3(x) - 3 \int \sec^3(x) (\sec^2(x) - 1) dx \\
= \tan(x) \sec^3(x) - 3 \int \sec^5(x) dx + 3 \int \sec^3(x) dx \\
= \tan(x) \sec^3(x) - 3I + \frac{3}{2} [\sec(x) \tan(x) + \ln(\sec(x) + \tan(x))].
\]

Therefore,

\[ I = \frac{1}{4} \tan(x) \sec^3(x) + \frac{3}{8} [\sec(x) \tan(x) + \ln(\sec(x) + \tan(x))] + C. \]
5. (10 points)

\[ \int \frac{2x^2 - x + 1}{(x - 2)(x^2 + 3)} \, dx \]

**Solution:** By partial fractions,

\[ \frac{2x^2 - x + 1}{(x - 2)(x^2 + 3)} = \frac{1}{x - 2} + \frac{x + 1}{x^2 + 3} \]

Therefore,

\[
\int \frac{2x^2 - x + 1}{(x - 2)(x^2 + 3)} \, dx = \int \frac{1}{x - 2} \, dx + \int \frac{x + 1}{x^2 + 3} \, dx \\
= \ln|x - 2| + \frac{1}{2} \int \frac{2x}{x^2 + 3} \, dx + \int \frac{1}{x^2 + 3} \, dx \\
= \ln|x - 2| + \frac{1}{2} \ln|x^2 + 3| + \frac{1}{\sqrt{3}} \arctan \left( \frac{x}{\sqrt{3}} \right) + C.
\]
6. (10 points) Find the volume of the solid obtained by rotating the region between \( y = -x^2 + 6x - 5 \) and \( y = 3x - 3 \) around \( y = -3 \).

**Solution:** The area of each cross-section is

\[
\pi(-x^2 + 6x - 5)^2 - \pi(3x - 3 + 3)^2 = \pi(x^4 - 12x^3 + 31x^2 - 24x + 4).
\]

The curves intersect at \( x = 1, 2 \). Therefore the volume is,

\[
\pi \int_1^2 (x^4 - 12x^3 + 31x^2 - 24x + 4) \, dx = \frac{23}{15}.
\]
7. Determine if the following integrals are convergent or divergent.

(a) (5 points)
\[ \int_{1}^{\infty} \frac{\sin \left( \frac{1}{x} \right)}{x^3 + 2x^2 + 3} \, dx \]

(b) (5 points)
\[ \int_{5}^{\infty} \frac{1}{(2x - 5)(3x + 6)} \, dx \]

**Solution:**

(a) Note that for \( x > 1 \), \( 0 \leq \sin(1/x) \leq 1 \). Moreover, since \( x^3 + 2x^2 + 3 \geq x^3 \geq 0 \), we have
\[
0 \leq \frac{\sin \left( \frac{1}{x} \right)}{x^3 + 2x^2 + 3} \leq \frac{1}{x^3}.
\]
Since the bigger one converges the integral itself is convergent by the comparison theorem.

(b) One can go ahead and integrate this by partial fractions. Here is an easier way: Note that for \( x \geq 5 \) we have \( 3x + 6 > 3x \) which implies \( 0 \leq \frac{1}{3x+6} \leq \frac{1}{3x} \). On the other hand, for \( x > 5 \) we also have \( 2x - 5 \geq x \geq 0 \). To see this note the following: \( y = 2x - 5 \) intersects \( y = x \) at the point \( x = 5 \). Since the slope of \( y = 2x - 5 \) is bigger than that of \( y = x \), for \( x \geq 5 \) the line \( y = 2x - 5 \) stays above the line \( y = x \). Therefore we have,
\[
0 \leq \frac{1}{(2x - 5)(3x + 6)} \leq \frac{1}{3x^2}.
\]
Since the bigger one is convergent by the \( p \)-test, our integral is also convergent by comparison theorem.