Name and UNI: ________________________________________________________________

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Instructions:

- There are 6 questions on this exam.
- In order to get full credit you only need to answer the first 5 questions correctly. The last question is bonus question.
- Please write your NAME and UNI on top of EVERY page.
- Unless otherwise is explicitly stated SHOW YOUR WORK in every question.
- Please write neatly, and put your final answer in a box.
- No calculators, cell phones, books, notebooks, notes or cheat sheets are allowed.
- Useful identities:
  \[
  \sin^2(\theta) + \cos^2(\theta) = 1, \quad \tan^2(\theta) + 1 = \sec^2(\theta)
  \]
  \[
  \sin(2\theta) = 2\sin(\theta)\cos(\theta), \quad \cos(2\theta) = 2\cos^2(\theta) - 1 = 1 - 2\sin^2(\theta)
  \]
- The roots of the equation \( ax^2 + bx + c = 0 \) are given by
  \[
  x_1, x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
  \]
1. Determine if the following statements are true (T) or false (F). You DO NOT have to justify your answer for this question.

(a) (1 point) If \( \lim_{n \to \infty} a_n = 0 \) then \( \sum_{n=1}^{\infty} a_n \) is convergent.

(b) (1 point) If \( a_n \neq 0 \) for every \( n \geq 1 \) and \( \sum_{n=1}^{\infty} a_n \) and \( \sum_{n=1}^{\infty} b_n \) are both convergent, then \( \sum_{n=1}^{\infty} a_n b_n \) is also convergent.

(c) (1 point) If \( \sum_{n=1}^{\infty} a_n 4^n \) is convergent then \( \sum_{n=1}^{\infty} a_n 4^n \) is divergent.

(d) (1 point) If \( \sum_{n=1}^{\infty} a_n \) is convergent and \( \sum_{n=1}^{\infty} b_n \) is divergent, then \( \sum_{n=1}^{\infty} a_n b_n \) is divergent.

(e) (1 point) If \( \{a_n\}_{n=1}^{\infty} \) is a sequence such that \( a_n > 0 \) for every \( n \geq 1 \), then \( \sum_{n=1}^{\infty} a_n n^p \) is divergent for every \( p < 1 \).

Solution:

(a) False. e.g. \( a_n = 1/n \).

(b) False. e.g. \( a_n = \frac{1}{n^2}, b_n = 0 \).

(c) False. e.g. \( a_n = \frac{1}{n^p} \).

(d) False. e.g. \( a_n = 0 \) and \( b_n = 1 \).

(e) False. e.g. \( a_n = \frac{1}{n^p} \) is convergent for \( p > 0 \).

(f) False. e.g. \( a_n = \frac{1}{n} \).

(g) True. \( \sum_{n=1}^{\infty} |a_n| \) convergent \( \Rightarrow |a_n| \to 0 \Rightarrow a_n \to 0 \).

(h) False. By the theorem on power series the power series can have only one of the 3 possibilities: converges only at a single point, converges on a bounded interval, converges for all values of \( x \).

(i) True. Since the interior of the interval of convergence is symmetric around the center we have \( c = \frac{-12+4}{2} = 4 \).

(j) True. By the ratio test.
2. In each part give an example of a series that satisfies the criterion. (You **NEED TO SHOW** that your example satisfies the properties. If you only write series without giving reason you **WILL NOT** get any credit.)

   (a) (3 points) A divergent series \( \sum_{n=1}^{\infty} a_n \) such that \( \sum_{n=1}^{\infty} a_n^2 \) is convergent.

   (b) (4 points) A convergent series \( \sum_{n=1}^{\infty} a_n \) such that \( \sum_{n=1}^{\infty} a_n^2 \) is divergent.

   (c) (5 points) A power series with interval of convergence \([-1, 7]\), where the series is **only conditionally convergent** at \( x = -1, 7 \).

**Solution:**

   (a) Take \( a_n = \frac{1}{n} \) satisfies the properties. \( \sum_{n=1}^{\infty} \frac{1}{n} \) is divergent because it is the harmonic series and since \( a_n^2 = \frac{1}{n^2} \), \( \sum_{n=1}^{\infty} a_n^2 \) converges by the \( p \)-test.

   (b) Take \( a_n = \frac{(-1)^n}{\sqrt{n}} \). Then \( \sum_{n=1}^{\infty} a_n \) converges by the alternating series test, however since \( a_n^2 = \frac{1}{n} \), \( \sum_{n=1}^{\infty} a_n^2 \) is divergent because it is the harmonic series.

   (c) Since the interval of convergence is symmetric around the center of the power series, we necessarily have to have \( c = \frac{-1+7}{2} = 3 \). This also implies that the radius has to be 4. Then an example of a series satisfying the conditions is \( \sum_{n=1}^{\infty} \frac{(-1)^n(x-3)^{2n}}{n4^n} \). By the ratio test we see that \( r = 4 \). Then checking the end points, we see that it converges at the both end points by the alternating series test.
3. Find the radius and interval of convergence of the following power series. Also determine if they are conditionally or absolutely convergent at the end points of the interval of convergence.

(a) (6 points)
\[ \sum_{n=1}^{\infty} \frac{x^{3n}}{n8^n} \]

(b) (4 points)
\[ \sum_{n=1}^{\infty} n \ln(n)x^n \]

**Solution:**

(a) By the ratio test we get
\[ \left| \frac{a_{n+1}}{a_n} \right| = \frac{nx^3}{8(n+1)} \to \frac{x^3}{8}. \]

Therefore \( R = 2 \). At the end point \( x = -2 \) we get the series \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \) which is convergent by the alternating series test. We also note that the convergence is conditional. At the end point \( x = -2 \) we get \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \) which is divergent. Therefore \( I = [-2, 2) \).

(b) By the ratio test
\[ \left| \frac{a_{n+1}}{a_n} \right| = \frac{(n+1) \ln(n+1) |x|}{n \ln(n)} \to |x|. \]

Therefore \( R = 1 \). At the end points \( x = \pm 1 \) the series is divergent by the divergence test. Hence \( I = (-1, 1) \).
4. Calculate the limit of the following sequences.

(a) (5 points)
\[ \lim_{n \to \infty} \left( 1 - \frac{12}{n} \right)^n \]

(b) (5 points)
\[ \lim_{n \to \infty} \frac{(-1)^n n^2 \sin(3n)}{n^2 + 2} \]

(c) (5 points)
\[ \lim_{n \to \infty} a_n, \]
where \( a_n \) is defined by
\[ a_n = 2 + \frac{1}{2 + a_{n-1}}, \quad a_0 = 1 \]

(You don’t have to show that the sequence \( \{a_n\}_{n=0}^\infty \) has a limit. You should simply assume that the limit exists and calculate its value.)

Solution:

(a) The limit is indeterminate of the type \( 1^\infty \). We take ln of both sides as usual.
\[
\lim_{n \to \infty} \ln \left( 1 - \frac{12}{n} \right)^n = \lim_{n \to \infty} n \ln \left( 1 - \frac{12}{n} \right) \\
= \lim_{n \to \infty} \frac{n \ln (1 - \frac{12}{n})}{1/n} \\
= \lim_{n \to \infty} \frac{12}{n} \left( 1 - \frac{12}{n} \right)^{-1} \\
= -12. 
\]
Therefore,
\[ \lim_{n \to \infty} \left( 1 - \frac{12}{n} \right)^n = e^{-12}. \]

(b) Since \( |(-1)^n n^2 \sin(3n)| \leq 1 \) we have \[\frac{1}{n^2 + 2} \leq \frac{(-1)^n n^2 \sin(3n)}{n^2 + 2} \leq \frac{1}{n^2 + 2}, \]
by the squeeze theorem we get that the limit is 0.

(c) Let the limit be \( L \). Then
\[ L = 2 + \frac{1}{2 + L} \Rightarrow L^2 = 5. \]
Since \( a_0 = 1 \) the terms of the series are all positive. Therefore the limit is \( \sqrt{5} \).
5. (10 points) Find the orthogonal trajectories to the family of curves given by

\[ y = e^{2k^3x} \quad \text{where} \quad k, x > 0. \]

(You do not have to solve for \( y \) in terms of \( x \) in the final answer.)

Solution:

\[ y' = 2k^3 e^{2k^3x} \Rightarrow y' = \frac{y \ln(y)}{x}. \]

Therefore the differential equation for the orthogonal trajectories is \( y' = \frac{-x}{y \ln(y)} \). Separating the variables and integrating by parts gives,

\[ \int y \ln(y) dy = - \int x dx \Rightarrow \frac{y^2 \ln(y)}{2} - \frac{y^2}{4} = - \frac{x^2}{2} + C. \]
6. (5 points (bonus)) Give an example of a pair of series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ such that $\lim_{n \to \infty} \frac{a_n}{b_n} = 1$, $\sum_{n=1}^{\infty} a_n$ is convergent $\sum_{n=1}^{\infty} b_n$ is divergent.

**Solution:** Take $a_n = \frac{(-1)^n}{\sqrt{n}}$ and $b_n = \frac{(-1)^n}{\sqrt{n}} + \frac{1}{n^{2/3}}$. 