Instructions:
• There are 9 questions on this exam.
• Please write your NAME and UNI on top of EVERY page.
• In order to get full credit you need to answer the first 8 questions correctly.
• The last question is a bonus question, and you do not have to answer it.
• Unless otherwise is explicitly stated SHOW YOUR WORK in every question.
• Please write neatly, and put your final answer in a box.
• No calculators, cell phones, books, notebooks, notes or cheat sheets are allowed.

\[
\sin^2(\theta) + \cos^2(\theta) = 1, \quad \tan^2(\theta) + 1 = \sec^2(\theta), \quad \sin(2\theta) = 2\sin(\theta)\cos(\theta)
\]

\[
\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = 2\cos^2(\theta) - 1 = 1 - 2\sin^2(\theta)
\]

\[
\sum_{n=A}^{B} r^n = \frac{r^{B+1} - r^A}{r - 1}
\]

\[
\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}
\]
1. Let \( C \) be the curve given parametrically by 
\[
C = \{ x = e^{4t}, y = 4te^{-4t} \mid t \in \mathbb{R} \}.
\]

(a) (3 points) Calculate \( \frac{dy}{dx} \).

(b) (2 points) Write the equation of the tangent line to \( C \) at \((1,0)\).

(c) (3 points) Calculate \( \frac{d^2y}{dx^2} \).

(d) (2 points) Find the points of intersection of \( C \) with the curve given by 
\[
C' = \{ x = e^{8t}, y = t^2e^{-8t} \mid t \in \mathbb{R} \}.
\]

**Solution:**

(a) 
\[
\frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx} = \frac{4e^{-4t} - 16te^{-4t}}{4e^{4t}} = \frac{1 - 4t}{e^{8t}}.
\]

(b) The value of \( t \) corresponding to the point \((1,0)\) is \( t = 0 \). By part (a) the slope at \( t = 0 \) is 1. Therefore the equation of the tangent line is 
\[
y = x - 1.
\]

(c) 
\[
\frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d}{dx} \left( \frac{1 - 4t}{e^{8t}} \right) = \frac{-4e^{8t} - 8(1 - 4t)e^{8t}}{e^{16t}/4e^{4t}} = \frac{8t - 3}{e^{12t}}.
\]

(d) Let \( t_1 \) denote the parameter for \( C_1 \) and \( t_2 \) denote the parameter for \( C_2 \). Then equating the \( x \)-coordinates we get 
\[
e^{4t_1} = e^{8t_2} \Rightarrow t_1 = 2t_2.
\]
Substituting this in the equation for \( y \)-coordinates we get 
\[
8t_2e^{-8t_2} = t_2^2e^{-8t_2} \Rightarrow 8t_2 = t_2^2 \Rightarrow t_2 = 0, 8.
\]
Therefore the two curves intersect at the points \((1,0)\) and \((e^{64}, 64e^{-64})\).
2. The aim of this question is to show that the series
\[ \sum_{n=1}^{\infty} \left( \sqrt[5]{5} - 1 \right) \]
diverges.

(a) (1 point) Using the identity \( a^b = e^{b \ln(a)} \), write \( \sqrt[5]{5} \) as \( e^{\frac{\ln(5)}{5}} \).

(b) (3 points) Using the Taylor expansion of \( e^x \) show that for any \( n \geq 1 \)
\[ \sqrt[5]{5} - 1 = \sum_{m=1}^{\infty} \frac{1}{m!} \left( \frac{\ln(5)}{n} \right)^m. \]

(c) (4 points) Using part (b) show that
\[ \lim_{n \to \infty} \left( \sqrt[5]{5} - 1 \right) = 0. \]

(d) (4 points) Using part (b) and the fact that \( 5 > e \), show that the series
\[ \sum_{n=1}^{\infty} \left( \sqrt[5]{5} - 1 \right) \]
is divergent. *Hint*: Comparison theorem.

**Solution:**

(a)
\[ \sqrt[5]{5} = 5^{\frac{1}{5}} = e^{\frac{\ln(5)}{5}}. \]

(b) Recall that the Taylor expansion of \( e^x \) is \( \sum_{m=0}^{\infty} \frac{x^m}{m!} \). This is convergent for every \( x \in \mathbb{R} \) so substituting \( x = \frac{\ln(5)}{n} \) we get
\[ \sqrt[5]{5} - 1 = \sum_{m=1}^{\infty} \frac{1}{m!} \left( \frac{\ln(5)}{n} \right)^m. \]

(c) Notice that in the series in part (b) for each \( m \geq 1 \) we have \( \frac{1}{m!} \frac{\ln(5)^m}{n^m} \). The limit of this as \( n \to \infty \) is 0. Therefore the overall limit is 0.

(d) Note first that since \( \ln(5) > 0 \)
\[ \sum_{m=1}^{\infty} \frac{1}{m!} \left( \frac{\ln(5)}{n} \right)^m > \frac{\ln(5)}{n}. \]

Furthermore since \( \ln(3) > 1 \) we have
\[ \frac{\ln(5)}{n} > \frac{1}{n}. \]

Therefore
\[ \sqrt[5]{5} - 1 > \frac{1}{n}. \]

Since both are positive, by comparison theorem
\[ \sum_{n=1}^{\infty} \left( \sqrt[5]{5} - 1 \right) > \sum_{n=1}^{\infty} \frac{1}{n}. \]

Therefore the series is divergent.
3. Let $p \in \mathbb{R}$ and consider the following series:

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln^p(n)}.$$

(a) (3 points) For any $p \in \mathbb{R}$ calculate the derivative of the function

$$f(x) = \frac{1}{x \ln^p(x)}.$$

(b) (2 points) Using part (a) show that $f(x)$ is decreasing for $x > e^{-p}$.

(c) (4 points) Show that for every $p \in \mathbb{R}$

$$\lim_{x \to \infty} f(x) = 0.$$

(d) (3 points) Using parts (b) and (c) show that the sequence given in the question is convergent for every $p \in \mathbb{R}$.

**Solution:**

(a)

$$f'(x) = \frac{- \ln^p(x) + p \ln^{p-1}(x)}{(x \ln^p(x))^2} = - \frac{\ln^{p-1}(x)(\ln(x) + p)}{(x \ln^p(x))^2}.$$

(b) If $x > e^{-p}$ then $\ln(x) > -p$ and therefore $\ln(x) + p > 0$. Hence if $x > e^{-p}$

$$f'(x) = - \frac{\ln^{p-1}(x)(\ln(x) + p)}{(x \ln^p(x))^2} < 0.$$

Therefore if $x > e^{-p}$ $f(x)$ is decreasing.

(c)

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{1}{x \ln^p(x)}.$$

If $p \geq 0$ then this is obviously 0. If $p < 0$ then $-p > 0$. Let $p_1 = -p$. Then observe that by L'Hospital’s rule we have

$$\lim_{x \to \infty} \frac{\ln^{p_1}(x)}{x} = \lim_{x \to \infty} \frac{p_1 \ln^{p_1-1}(x)}{x} = \lim_{x \to \infty} \frac{p_1(p_1 - 1) \ln^{p_1-2}(x)}{x} = \lim_{x \to \infty} \frac{p_1(p_1 - 1)(p_1 - 2) \ln^{p_1-3}(x)}{x},$$

where we apply L'Hospital’s rule at each step. Since $p_1 > 0$ by assumption, after finitely many steps (namely the first integer that is greater than or equal to $p_1$ steps) the exponent of the $\ln$ will become $\leq 0$. Therefore the limit is 0.

(d) By parts (b) and (c) we can use the alternating series test, which shows that the series converges for every $p \in \mathbb{R}$. 
4. Let \(a, b \geq 1\) denote a positive integers and consider the following series

\[
\sum_{n=0}^{\infty} \frac{(n!)^a}{(bn)!}.
\]

**Hint:** Ratio test.

(a) (1 point) Show that for \(a = 1, b = 1\) the series is divergent.

(b) (4 points) Show that for \(a = 1, b \geq 2\) the series is convergent.

(c) (5 points) Show that for \(a \geq 2\), the series is convergent if \(b \geq a\) and divergent if \(b < a\).

**Solution:**

(a) For \(a = b = 1\) the series is \(\sum_{n=0}^{\infty} 1\) which is divergent by the divergence test.

(b) For \(a = 1, b \geq 2\) we are considering the series

\[
\sum_{n=0}^{\infty} \frac{n!}{(bn)!}.
\]

Using the ratio test we get,

\[
\lim_{n \to \infty} \left| \frac{(n+1)!}{(b(n+1))!} \right| \left| \frac{n!}{(bn)!} \right| = \lim_{n \to \infty} \left| \frac{n+1}{(bn+b)(bn+b-1) \cdots (bn+1)} \right| = 0.
\]

Therefore the series is absolutely convergent.

(c) We again use the ratio test.

\[
\lim_{n \to \infty} \left| \frac{(n+1)!^a}{(b(n+1))!} \right| \left| \frac{(n!)^a}{(bn)!} \right| = \lim_{n \to \infty} \left| \frac{(n+1)^a}{(bn+b)(bn+b-1) \cdots (bn+1)} \right|.
\]

Note that the degree of the polynomial in the numerator is \(a\) and the one in the denominator is \(b\). Therefore we immediately get that the limit is \(\infty\) if \(a > 0\) and \(0\) if \(b < a\), and hence the series converges absolutely if \(a < b\) and diverges if \(b > a\). The only remaining case is when \(a = b\). In that case the limit is the ratio of the leading coefficients, which is \(\frac{1}{b^a}\). Since we are assuming \(a \geq 2\) and \(a = b\), we necessarily have \(b \geq 2\) and hence \(b^a \geq 4\) which shows that the limit is \(\leq \frac{1}{4}\). Hence by the ratio test the series is again absolutely convergent.
5. Give an example of a power series that satisfies the following properties. (You need to verify that the series satisfies the conditions to get full credit!)

(a) (3 points) Its interval of convergence is $(-5, 3]$ and its derivative’s interval of convergence is $(-5, 3)$.

(b) (4 points) Its interval of convergence is $[-5, 3]$ and its derivative’s interval of convergence is also $[-5, 3]$.

(c) (5 points) Its interval of convergence is $(-5, 3]$ and its derivative’s interval of convergence is also $(-5, 3]$. Hint: You may find it useful to think about conditional convergence. Hint-2: You may also find it useful to think about $\ln(n)$ as well as $n$.

**Solution:**

(a) The interval of convergence being $(-5, 3]$ implies that the series should be centered at $-1$ with radius 4. An example is $\sum_{n=1}^{\infty} \frac{(-1)^n(x+1)^n}{n^4}$. By the ratio test this is absolutely convergent in $(-5, 3)$. At the boundaries we get the harmonic series at $x = -5$ and $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ at $x = 3$. Therefore the interval is $(-5, 3]$. The derivative on the other hand is given by $\sum_{n=1}^{\infty} \frac{(-1)^n(x+1)^n}{n^4}$ which is divergent at both end points (note that the radius does not change by the theorem on differentiation of power series), hence has interval of convergence $(-5, 3)$.

(b) For this one take, for example, $\sum_{n=1}^{\infty} \frac{(x+1)^n}{n^4}$. This has radius 4 by the ratio test and is absolutely convergent at the end points so it has interval of convergence $[-5, 3]$. Moreover, term by term differentiation gives that the derivative is $\sum_{n=1}^{\infty} \frac{(x+1)^n}{n^4}$ which is still absolutely convergent at the end points and hence has interval of convergence $[-5, 3]$.

(c) This part is trickier. An example is given by $\sum_{n=2}^{\infty} \frac{(-1)^n(x+1)^n}{n \ln(n) 4^n}$. This is only conditionally convergent at $x = 3$ and divergent at $x = -5$, which shows that the interval of convergence is $(-5, 3]$. On the other hand the derivative is $\sum_{n=2}^{\infty} \frac{(-1)^n(x+1)^n}{\ln(n) 4^n}$ which is conditionally convergent at $x = 3$, by the alternating series test, and is divergent at $x = -5$. Hence still has interval of convergence $(-5, 3]$
6. Consider 
\[ f(x) = e^{x^2}. \]

(a) (2 points) Using the Taylor expansion of \( e^x \) find the Taylor expansion of \( f(x) \).

(b) (4 points) Using the expansion you found in part (a) show that for any integer \( n \geq 0 \) the \( 2n \)’th derivative of \( f \) is given by 
\[ f^{(2n)}(0) = \frac{(2n)!}{n!}. \]

**Solution:**

(a) Substituting \( x^2 \) in the Taylor expansion of \( e^x \) we get 
\[ f(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}. \]

(b) By the definition of Taylor series, 
\[ f(x) = \sum_{m=0}^{\infty} \frac{f^{(m)}(0)}{m!} x^m. \]

On the other hand by part (a) we have 
\[ \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}. \]

Equating the coefficients of powers of \( x \) we see that \( \frac{f^{(2m)}(0)}{(2m)!} = \frac{1}{m!} \). Therefore we get that 
\[ f^{(2m)}(0) = \frac{(2m)!}{m!}. \]
7. Consider the following first order non-linear differential equation.

\[ x \frac{dy}{dx} + y^2 \ln(x) = y. \]

(a) (4 points) Use the substitution \( u = \frac{1}{y} \) to transform the equation into a linear one.

(b) (6 points) Find the general solution to the equation you found in part (a).

(c) (2 points) Using the solution you found in part (b) determine the general solution of the original equation.

**Solution:**

(a) Since \( u = \frac{1}{y} \), \( \frac{du}{dx} = -\frac{1}{y^2} \frac{dy}{dx} \). Substituting this into the equation then gives

\[ -x \frac{du}{dx} + \frac{x \ln(x)}{u^2} = \frac{1}{u} \Rightarrow \frac{du}{dx} + \frac{u}{x} = \frac{\ln(x)}{x}. \]

(b) An integrating factor is \( x \) for this equation. Multiplying both sides by \( x \) gives

\[ (xu)' = \ln(x) \Rightarrow u = \ln(x) - 1 + \frac{C}{x}. \]

(c) Substituting \( u = \frac{1}{y} \) gives

\[ y = \frac{1}{\ln(x) - 1 + \frac{C}{x}}. \]
8. Let $a_0 = 2, a_1 = 4$ and for every $n \geq 2$ let $a_n$ be defined by

$$a_n = \frac{1}{2}(a_{n-1} + a_{n-2}).$$

In this question we will calculate

$$\lim_{n \to \infty} a_n.$$

We will do this by using an auxiliary sequence $b_n$ defined for $n \geq 0$ as follows:

$$b_n = a_{n+1} - a_n.$$

(a) (3 points) Show that for every $n \geq 0$

$$b_{n+1} = -\frac{b_n}{2}.$$

(b) (2 points) Using the result in part (a) deduce that

$$b_n = \frac{(-1)^n}{2^{n-1}}.$$

(c) (3 points) Using the definition of $b_n$ show that for every $n \geq 1$

$$a_n = 2 + \sum_{m=0}^{n-1} b_m.$$

(d) (4 points) Using part (c) and the geometric sum formula show that

$$\lim_{n \to \infty} a_n = \frac{10}{3}.$$

**Solution:**

(a) By the definition of $b_n$ and the recurrence for $a_n$ we get

$$b_{n+1} = a_{n+2} - a_{n+1}$$

$$= \frac{1}{2}(a_{n+1} - a_n) - a_{n+1}$$

$$= -\frac{1}{2}(a_{n+1} - a_n)$$

$$= -\frac{b_n}{2}.$$

(b)

$$b_n = -\frac{1}{2}b_{n-1} = \frac{(-1)^2}{2^2}b_{n-2} = \cdots = \frac{(-1)^n}{2^n}b_0.$$

Note that $b_0 = a_1 - a_0 = 2$. Therefore, $b_n = \frac{(-1)^n}{2^n}.$

(c)

$$a_n = a_n - a_{n-1} + a_{n-1} + a_{n-2} - a_{n-2} + a_{n-3} - a_{n-3} + \cdots - a_1 + a_1 - a_0 + a_0$$

$$= b_{n-1} + b_{n-2} + b_{n-3} + \cdots + b_1 + b_0 + a_0$$

$$= \sum_{m=0}^{n-1} b_m + 2.$$
(d) By part (c) we have

\[ a_n = 2 + \sum_{m=0}^{n-1} b_m. \]

Therefore,

\[ \lim_{n \to \infty} a_n = 2 + \sum_{m=0}^{\infty} b_m. \]

By part (b) we know that \( b_n = \frac{(-1)^n}{2^{n-1}} \). Therefore,

\[
\begin{align*}
\lim_{n \to \infty} a_n &= 2 + \sum_{m=0}^{\infty} b_m \\
&= 2 + \sum_{m=0}^{\infty} \frac{(-1)^m}{2^{m-1}} \\
&= 2 + \frac{1}{1 + \frac{1}{2}} \\
&= \frac{10}{3}.
\end{align*}
\]
9. (6 points (bonus)) Using the infinite product expansion

\[
\frac{\sin(\pi x)}{\pi x} = (1 - x^2) \left(1 - \frac{x^2}{4}\right) \left(1 - \frac{x^2}{9}\right) \left(1 - \frac{x^2}{16}\right) \cdots
\]

show that

\[
\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.
\]

*Hint:* Think of Taylor series.

**Solution:**

The idea is to compare the coefficient of \( x^2 \) in the Taylor expansion of the left hand side with the corresponding coefficient on the right hand side. First of all, on the right hand side, in order to get \( x^2 \), we have to choose 1 form all but one of the factors, and choose \( \frac{x^2}{n^2} \) on exactly one factor. Therefore the coefficient of \( x^2 \) on the right hand side is

\[
-\sum_{n=1}^{\infty} \frac{1}{n^2}.
\]

On the other hand since the Taylor series of \( \sin(x) \) is \( \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \), we see that

\[
\frac{\sin(\pi x)}{\pi x} = \frac{1}{\pi x} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (\pi x)^{2n+1}.
\]

Now we see that the coefficient of \( x^2 \), which corresponds to the term \( n = 1 \), in this expression is

\[
\frac{-\pi^2}{6}.
\]

*Side remark:* This observation by made first by the famous mathematician Leonhard Euler in the early 18th century.