Cayley Graphs of Free Groups

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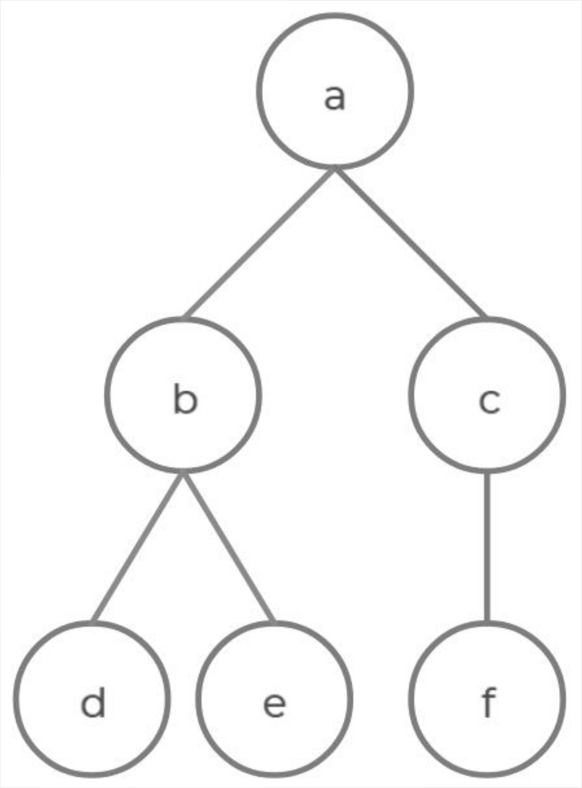
Trees

Definition 3.1.8 (Tree)

A *tree* is a connected graph that does not contain any cycles.

A Graph that does not contain any cycles is a *forest*; so, a tree is the same as a connected forest.

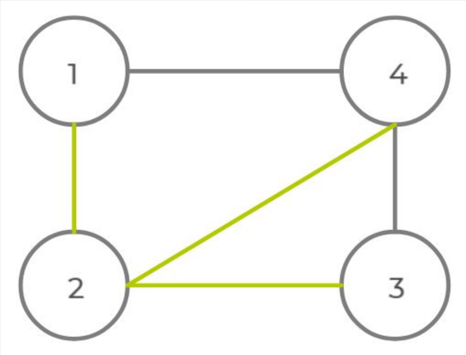
X = ({a, b, c, d, e, f}, {{a, b}, {a, c}, {b, d}, {b, e}, {c, f}})



Characterising Trees

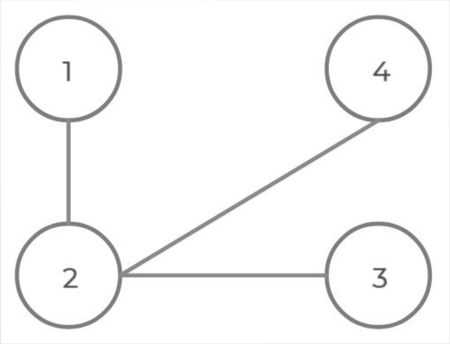
Proposition 3.1.10 (Characterising Trees)

A graph is a tree IFF for every pair of vertices there exists exactly one path connecting these vertices



Definition 3.1.11 (Spanning Trees)

A *spanning tree* of a graph *X* is a subgraph of *X* that is a tree and contains all vertices of *X*.
A *subgraph* of a graph (V, E) is a graph (V', E') with $V' \subset V$ and $E' \subset E$



Cayley Graphs

Definition 3.2.1

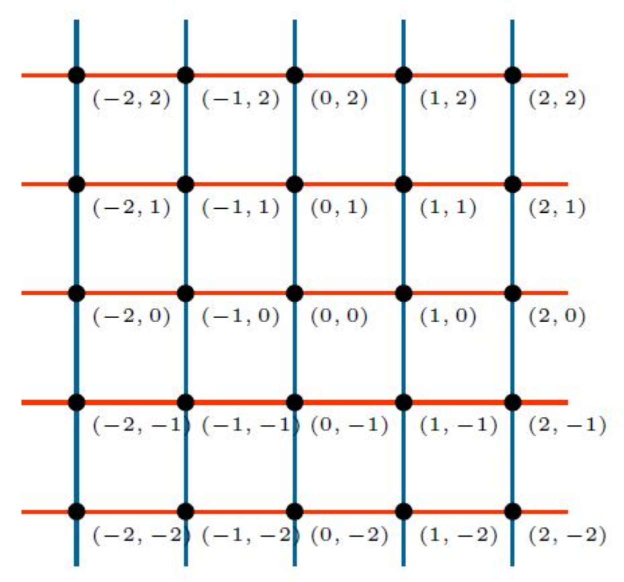
Let G be a group and $S \subset G$. Then the Cayley graph of G with respect to the generating set S is the graph Cay(G,S) whose

- Set of vertices is G
- Set of edges is

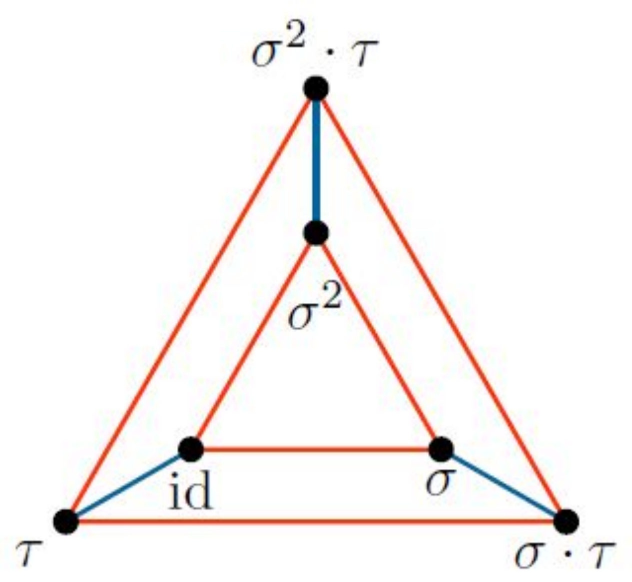
$$\{\{g, g \square s\} \mid g \in G, s \in (S \cup S^{-1}) \setminus \{e\}\}$$

That is, two vertices in a Cayley graph are adjacent IFF they differ by right multiplication by an (inverse of an) element of the generating set in question.
By definition, the Cayley graph with respect to a generating set S coincides with the Cayley graphs for S^{-1} and for $S \cup S^{-1}$.

Examples of Cayley Graphs



Cay($\boldsymbol{Z}^2$, {(1, 0),(0, 1)})



Cay(S_3, $\{\tau, \sigma\}$)

Cayley Graphs of Free Groups (3.3)

Theorem 3.3.1 Cayley Graphs of Free Groups

Let F be a free group, freely generated by $S \subset F$. Then the corresponding Cayley graph Cay(F, S) is a tree.

The converse is *not* generally true

Example

Non-free Groups with Cayley Trees

- Cayley graph Cay($\boldsymbol{Z}$/2, [1]) consists of two vertices joined by an edge. An example of a tree, but not a free group.
- The Cayley graph Cay($\boldsymbol{Z}$, {-1, 1}) coincides with Cay($\boldsymbol{Z}$, {1}), which is a tree. But {-1, 1} is not a free generating set of $\boldsymbol{Z}$.

Cayley Graphs of Free Groups

Theorem 3.3.3 Cayley Trees and Free Groups

Let G be a group, let $S \subset G$ be a generating set satisfying $s \square t \neq e$ $\forall$ s, t. If the Cayley graph Cay(G, S) is a tree, then S is a free generating set of G.

In order to dive into a formal proof, we must first describe free groups in terms of reduced words.

We must solve the word problem of G with respect to S.

Free Groups & Reduced Words

Definition
Word

Let S be a set and F(S) be a group freely generated by S. A *word* w in S is a finite sequence of elements written as

$$w = s_1 \ldots s_n \text{ where } s \in S \cup S^{-1}$$

We define the *length* of the word w as n denoted as $|w| = n$. Note that the empty word e is the neutral element whose length $|e| = 0$.

The set of all elements $s \in S$ along with their inverses s^{-1} where $s \in S$ compose the alphabet of $F(S)$.

Free Groups & Reduced Words

Examples of Words

$S \cup S^{-1} = \{a, b, c, a^{-1}, b^{-1}, c^{-1}\}$

$$w = a \cdot a^{-1} = \varepsilon$$

$$w = b \cdot c^{-1} \cdot a \cdot a^{-1} \cdot b^{-1} = b \cdot c^{-1} \cdot b^{-1}$$

$F(S) = \mathbf{Z}^{+}$ $\quad S = \{1\}$ $\quad S^{-1} = \{-1\}$ $\quad S \cup S^{-1} = \{-1, 1\}$

$$w = e = -1 \cdot 1 = 0$$

$$w = 1 \cdot 1 - 1 \cdot 1 \cdot -1 \cdot 1 \cdot 1 = 3 = 1 \cdot 1 \cdot 1$$

Free Groups & Reduced Words

Definition 3.3.4
Reduced Word

Let S be a set, and let $(S \cup S^{-1})^*$ be the set of words of S and formal inverses of elements of S.

- Let $n \in \mathbb{N}$ and let $s_1,\ldots, s_n \in S \cup S^{-1}$. The word $s_1\ldots s_n$ is *reduced* if

$$s_{j+1} \neq s_j^{-1} \text{ and } s^{-1}_{j+1} \neq s_j$$

holds for all $j \in \{1,\ldots, n-1\}$

- We write $F_{red}(S)$ for the set of reduced words in $(S \cup S^{-1})^*$

Free Groups & Reduced Words

Proposition 3.3.5
Free Groups via Reduced Words

Let S be a set

1. The set $F_{red}(S)$ of reduced words over $S \cup S^{-1}$ forms a group with respect to the composition $F_{red}(S) \times F_{red}(S) \to F_{red}(S)$ given by

$$(s_1 \dots s_n,\ s_{n+1} \dots s_m) \mapsto (s_1 \dots s_{n-r} s_{n+1+r} \dots s_{n+m})$$

   where $s_1 \dots s_n$ and $s_{n+1} \dots s_m$ are in $F_{red}(S)$ (with $s_1 \dots s_n \in S \cup S^{-1}$) and

$$r := \max\{k \in \{0,\dots, \min(n, m-1)\} \mid \forall\ j \in \{0,\dots,k-1\}\ s_{n-j} = s^{-1}_{n+1+j} \vee s^{-1}_{n-j} = s_{n+1+j}\}$$

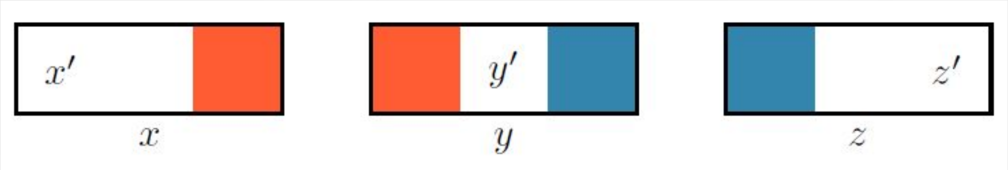
   IOW, the composition of reduced words is given by

   1. Concatenating the words
   2. Reducing maximally at the concatenation positions

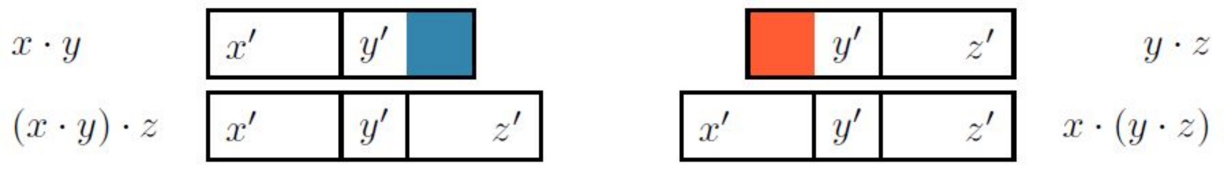
2. The group $F_{red}(S)$ is freely generated by S

Free Groups & Reduced Words

So far, our construction is well-defined. With the composition of two reduced words, we have a word composed in reduced form by construction. In our construction, the empty word ε, itself a reduced word is the neutral element.



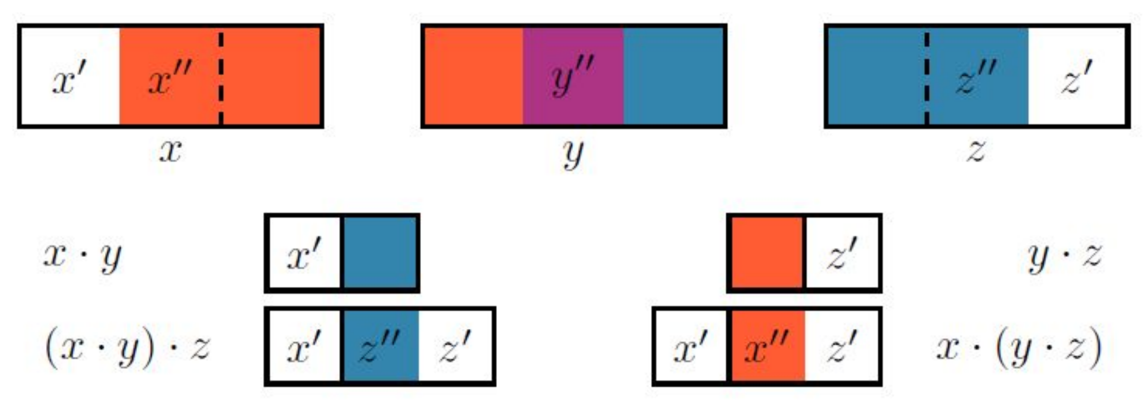
Associativity: Let $x, y, z \in F_{red}(S)$; we want to show that $(x \cdot y) \cdot z = x \cdot (y \cdot z)$. By definition, when composing two words, we have to remove the maximal reduction area where the two words meet.



If the reduction areas of x, y and y, z have no intersection in y, then clearly $(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$y^{-1}zxx^{-1}y \longrightarrow y^{-1}zy.$

Free Groups & Reduced Words



If the reduction areas of *x,y* and *y, z* have a non-trivial intersection *y"* in *y*, then the equality (*x·y*)·*z*=*x*·(*y·z*) follows by carefully inspecting the reduction areas in *x* and *z* and the neighboring regions; because of the overlap in *y"*, we know that *x"* and *z"* coincide (they are both inverse of *y"*).

Free Groups & Reduced Words

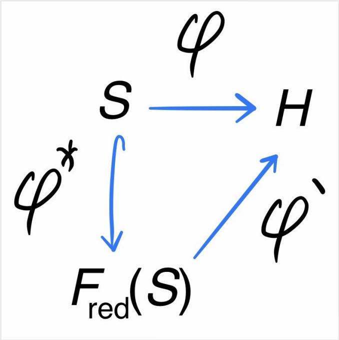
Establishing the Presence of the Universal Property

We show that S is a free generating set of $F_{red}(S)$ by verifying that the *universal property* is satisfied:

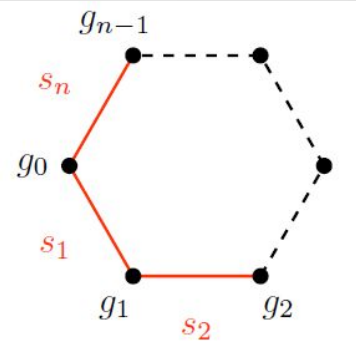
Let H be a group and let $\varphi: S \to H$ by a map. We can see that the following is a group homomorphism.

$$\varphi' := \varphi^* | F_{red}(S) : F_{red}(S) \to H$$

Clearly, $\varphi'|s = \varphi$; because S generated $F_{red}(S)$, it follows that φ' is the only such homomorphism. Hence, $F_{red}(S)$ is freely generated by S.



Free Groups → Trees



Proof of Theorem 3.3.1
Cayley Graphs of Free Groups

Suppose F is freely generated by S. By the last proposition, F is isomorphic to $F_{red}(S)$ via an isomorphism that is the identity on S; we can then assume F is $F_{red}(S)$.

Let's assume for contradiction that Cay(F, S) contains a cycle $g_0, \ldots g_{n-1}$ of length n with $n \geq 3$; specifically,

$$s_{j+1} := g_{j+1} \cdot g_{j+1}^{-1} \in S \cup S^{-1}, \text{ for all } j \in \{0, \ldots, n\text{-}2\}, \text{ as well as } s_n := g_0 \cdot g_{n-1}^{-1} \in S \cup S^{-1}$$

Because the vertices are distinct, the word $s_0 \ldots s_{n-1}$ is reduced; on the other hand, we obtain

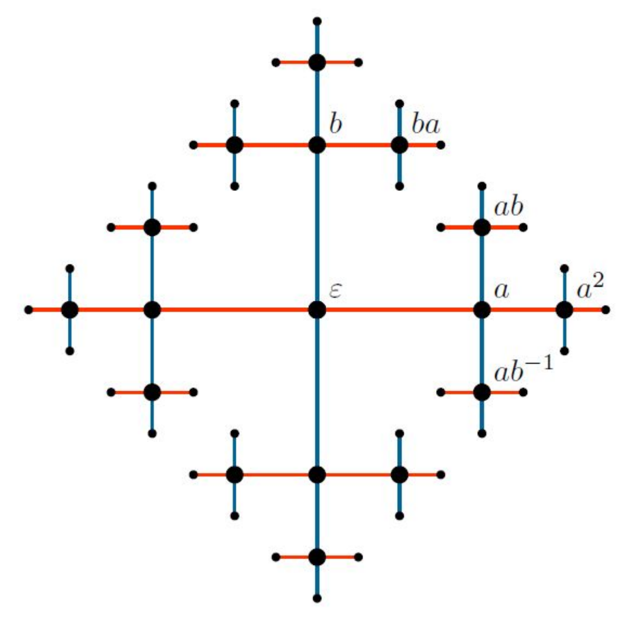
$$s_n \ldots s_1 = g_0 \cdot g_{n-1}^{-1} \cdots g_2 \cdot g_1^{-1} g_1 \cdot g_0^{-1} = e = \boldsymbol{\varepsilon}$$

In $F = F_{red}(S)$, which is impossible. Therefore, Cay(F, S) cannot contain any cycles, so Cay(F, S) is a tree

Free Groups → Trees

Example
Cayley graph of the free group of rank 2

Let *S* be a set consisting of two different elements *a* and *b*. Then the corresponding Cayley graph Cay(*F*(*S*), {*a*, *b*}) is a regular tree whose vertices have exactly four neighbors.



Trees → Free Groups

Proof of Theorem 3.3.3
Cayley Trees and Free Groups

Let G be a group and let $S \subset G$ be a generating set satisfying $s \cdot t \neq e \ \forall \ s, t \in S$ and s.t. The corresponding Cayley graph Cay(G, S) is a tree. To show that S is a free generating set of G, we just need to show that G is isomorphic to $F_{red}(S)$ via an isomorphism that is the identity on S.

Because $F_{red}(S)$ is freely generated by S, the universal property of free groups provides us with a group homomorphism $\varphi: F_{red}(S) \to G$ that is the identity on S. As S generates G, it follows that φ is surjective.

Assume for contradiction that φ is not injective. Let $s_1 ... s_n \in F_{red}(S) \setminus \{\boldsymbol{\varepsilon}\}$ with $s_1,...,s_n \in S \cup S^{-1}$ be an element of minimal length that is mapped to e by φ. We consider:

- Because $\varphi|s = \mathrm{id}_s$ is injective, it follows that $n > 1$
- If n=2, then it would follow that

  $$e = \varphi(s_1 \cdot s_2) = \varphi(s_1) \cdot \varphi(s_2) = s_1 \cdot s_2$$

  In G contradicting that $s_1 ... s_n$ is reduced and that $s \cdot t \neq e$ in $G \ \forall \ s, t \in S$

Trees → Free Groups

Proof of Theorem 3.3.3
Cayley Trees and Free Groups

- If $n \geq 3$, we consider the sequence $g_0,\dots,g_{n-1}$ of elements of G inductively by $g_0 := e$ and

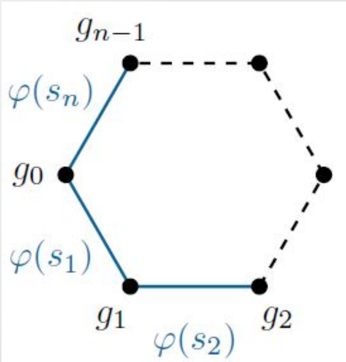
  $g_{j+1} := g_j \cdot s_{j+1}$

  For all $j \in \{0,\dots, n-2\}$. The sequence $g_0,\dots,g_{n-1}$ is a cycle in Cay(G, S) because by minimality of the word $s_1 \dots s_n$, the elements $g_0,\dots,g_{n-1}$ are distinct; moreover, Cay(G, S) contains the edges $\{g_0, g_1\},\dots,$ $\{g_{n-2}, g_{n-1}\}$, and the edge

  $$\{g_{n-1}, g_0\} = \{s_1 \cdot s_2 \cdots s_{n-1}, e\}$$
  $$= \{s_1 \cdot s_2 \cdots s_{n-1}, s_1 \cdot s_2 \cdots s_n\}$$

  However, this contradicts the hypothesis that Cay(G, S) is a tree.

  Hence φ: $F_{red}(S) \to G$ is injective.



Thank you!