COLUMBIA UNIVERSITY
IN THE CITY OF NEW YORK

Intro to Modern Algebra I
Math GU4041
New York, 2021/03/10

Exercise Sheet 8

Products and quotients

Exercise 1. Consider the subset

$$SL^{\pm}(n,\mathbb{R}) = \{\ A \in GL(n,\mathbb{R})\ \mid\ \det(A) = \pm 1\ \} \subset GL(n,\mathbb{R})\,.$$

(a) Prove that

$$SL^{\pm}(n,\mathbb{R}) \lhd GL(n,\mathbb{R})\,.$$

(b) Prove that for all $n \geq 1$,

$$GL(n,\mathbb{R}) \simeq SL^{\pm}(n,\mathbb{R}) \times (\mathbb{R}_{>0},\cdot)\,.$$

(c) If n is odd, prove that

$$GL(n,\mathbb{R}) \simeq SL(n,\mathbb{R}) \times (\mathbb{R}\setminus\{0\},\cdot)\,.$$

Exercise 2. Prove that

$$[G,G] \lhd G\,,$$

where $[G,G]$ is the commutator subgroup of G.

Exercise 3. Prove the following statements.

(a) If $H \lhd G$, then

$$\forall a \in G,\ aHa^{-1} < G\,.$$

(b) $H \lhd G \Leftrightarrow \forall a \in G,\ aHa^{-1} = H\,.$

(c) $H \lhd G \Leftrightarrow \forall a \in G,\ aH = Ha\,.$

Exercise 4. Let G be a group and $H < G$. Consider the following relation on G:

$$a \sim b \Leftrightarrow ab^{-1} \in H\,.$$

(a) Prove that this is an equivalence relation.

(b) Prove that the equivalence classes are the right cosets of H.

Exercise 5. Let $G = (\mathbb{C} \setminus \{0\}, \cdot)$. Consider the subgroups $\mathbb{S}^1$ and $\mathbb{R}_{>0}$. Draw a picture of the cosets of $\mathbb{S}^1$, and a picture of the cosets of $\mathbb{R}_{>0}$.

Exercise 6. Consider $\mathbb{Z}^2 \lhd \mathbb{R}^2$. Prove that

$$\mathbb{R}^2/\mathbb{Z}^2 \simeq \mathbb{S}^1 \times \mathbb{S}^1 .$$

Exercise 7 (Fundamental Theorem of Homomorphism)**.** Let $\varphi : G \to G'$ be a homomorphism, and let $N \lhd G$. Assume that $N \subset \ker \varphi$, and denote by π the quotient homomorphism, $\pi : G \to G/N$. Prove that there exists a unique homomorphism $f : G/N \to G'$ such that $\varphi = f \circ \pi$.

(Hint: the proof is the same as the proof of the First Isomorphism Theorem given in class. Just adapt it to this more general case.)

Exercise 8 (Conjugation of isometries)**.** Consider the subgroups defined in Homework 05, Exercise 7 and 8:

$$T = \left\{ \begin{pmatrix} 1 & 0 & v_x \\ 0 & 1 & v_y \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; v_x, v_y \in \mathbb{R} \right\} < \mathrm{Isom}(\mathbb{R}^2) \,,$$

consisting of translations and the identity, and

$$O = \left\{ \begin{pmatrix} \multicolumn{2}{c}{\multirow{2}{*}{A}} & 0 \\ & & 0 \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; A \in O(2) \right\} < \mathrm{Isom}(\mathbb{R}^2) \,,$$

consisting of the isometries that fix the origin.

(a) Let $t \in T$ be the translation by the vector $\begin{pmatrix} v_x \\ v_y \end{pmatrix}$. Given an $o \in O$, find a fixed point of the element tot^{-1}.
(Hint: no need to do computations with matrices, you can find it geometrically.)

(b) Given $t \in T$, describe the subgroup tOt^{-1}, and prove that O is not normal.

(c) For a general $x \in \mathrm{Isom}(\mathbb{R}^2)$ and $t \in T$, compute xtx^{-1}.

(d) Prove that $T \lhd \mathrm{Isom}(\mathbb{R}^2)$.