COLUMBIA UNIVERSITY
IN THE CITY OF NEW YORK

Intro to Modern Algebra I
Math GU4041
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EXERCISE SHEET 7

Homomorphisms

Exercise 1. For $\theta \in \mathbb{R}$, define the complex number

$$e^{i\theta} = \cos\theta + i\sin\theta\,.$$

Using trigonometric identities, prove that the map

$$\varphi : (\mathbb{R}, +) \longrightarrow (\mathbb{C} \setminus \{0\}, \cdot)$$

defined by

$$\varphi(\theta) = e^{i2\pi\theta}\,,$$

is a group homomorphism, and compute its kernel and image.

Exercise 2. Given a group G, define the following relation for $a, b \in G$:

$$a \sim b \;\Leftrightarrow\; \exists c \in G \text{ s.t. } b = cac^{-1}\,.$$

Prove that this relation is an equivalence relation.

Exercise 3. Given a group G, for $a, b \in G$ prove that

$$o(b) = o(aba^{-1})\,.$$

Exercise 4 (Conjugation and automorphisms)**.** Let G be a group.

(a) For $a \in G$ prove that the map

$$\varphi_a : G \to G\,,$$

defined by

$$\varphi_a(g) = aga^{-1}\,,$$

is an automorphism of G.

(b) Prove that the map

$$\varphi : G \to \mathrm{Aut}(G)\,,$$

sending $a \in G$ to φ_a is a group homomorphism.

(c) Prove that $\ker(\varphi) = Z(G)$. This implies that $Z(G) \lhd G$. (Note: $Z(G)$ was defined in HW06, Exercise 3).

(d) Prove that, for every automorphism $\psi \in \mathrm{Aut}(G)$,

$$\psi\varphi_a\psi^{-1} = \varphi_{\psi(a)}\,.$$

(e) Prove that

$$\varphi(G) \lhd \mathrm{Aut}(G)\,.$$

Exercise 5. Let $G_1, \ldots, G_n$ be groups. Prove that the cartesian product

$$G_1 \times \cdots \times G_n$$

is a group when the product is defined component-wise.

Exercise 6. Prove that $\mathbb{Z}_2 \times \mathbb{Z}_3 \simeq \mathbb{Z}_6$.

Exercise 7. Prove that $\mathbb{S}^1 \times \mathbb{R} \simeq (\mathbb{C} \setminus \{0\}, \cdot)$.

Exercise 8. Let G be a group and $g \in G$. Prove that there exists a unique homomorphism

$$\varphi : \mathbb{Z} \to G$$

such that $\varphi(1) = g$. Then compute $\ker \varphi$ (it depends on $o(g)$) and $\varphi(\mathbb{Z})$.

Exercise 9. Let G be an Abelian group and $g_1, g_2 \in G$. Prove that there exists a unique homomorphism

$$\varphi : \mathbb{Z}^2 \to G$$

such that $\varphi((1,0)) = g_1$, $\varphi((0,1)) = g_2$.

Exercise 10. Prove that $(\mathbb{Q}, +) \not\simeq (\mathbb{Q}_{>0}, \cdot)$.
(Hint: Prove that no homomorphism $\mathbb{Z}^2 \to (\mathbb{Q}, +)$ can be 1-1. Find a 1-1 homomorphism $\mathbb{Z}^2 \to (\mathbb{Q}_{>0}, \cdot)$.)
(Remark: This contrasts with the isomorphism $(\mathbb{R}, +) \simeq (\mathbb{R}_{>0}, \cdot)$).