Intro to Modern Algebra I
Math GU4041
New York, 2021/02/17

EXERCISE SHEET 6

Subgroups

Exercise 1. Let G be a group such that for all $a \in G$, $a^2 = e$. Prove that G is Abelian.

Exercise 2. Let G be a finite group of even cardinality. Prove that there exists $a \in G$ such that $a \neq e$ and $a^2 = e$. In other words, a group of even cardinality contains an element of order 2.
(Hint: use the fact that $(a^{-1})^{-1} = a$.)

Exercise 3. Given a group G, the **center** of G is defined as the subset

$$Z(G) = \{\ z \in G \ \mid\ \forall x \in G,\ zx = xz\ \}\,.$$

of the elements that commute with every element of G. Prove that $Z(G) < G$.

Exercise 4. Given a group G, and $a \in G$, the **centralizer** of a in G is defined as the subset

$$C(a) = \{\ c \in G \ \mid\ ca = ac\ \}\,.$$

of the elements that commute with a. Prove that $C(a) < G$.

Exercise 5. Let $m, n \in \mathbb{Z}$. Prove that

$$\langle m, n \rangle = \langle (m, n) \rangle\ .$$

Exercise 6. Classify all the subgroups of $(\mathbb{Z}, +)$. In order to do so, first prove that every subgroup of $\mathbb{Z}$ is cyclic.
(Hint: Given $A < \mathbb{Z}$, consider the smallest positive element of A.)

Exercise 7. For $r, s \in \mathbb{Q}$, prove that $\langle r, s \rangle$ is cyclic. Then, prove that every finitely generated subgroup of $(\mathbb{Q}, +)$ is cyclic. Conclude that $(\mathbb{Q}, +)$ is not finitely generated.

Exercise 8. Prove that $\left\langle 1, \sqrt{2} \right\rangle < (\mathbb{R}, +)$ is not cyclic.
(Hint: by contradiction: if $\left\langle 1, \sqrt{2} \right\rangle = \langle x \rangle$, then ...)