COLUMBIA UNIVERSITY
IN THE CITY OF NEW YORK

EXERCISE SHEET 1

Numbers

Exercise 1. Prove that, for all $n \in \mathbb{N}$,

$$\sum_{i=0}^{2n} i^2 = \frac{n(2n+1)(4n+1)}{3} \,.$$

Exercise 2. Prove that, for all $n \in \mathbb{N}$ with $n \geq 7$,

$$n! > 3^n \,.$$

Exercise 3. Consider the sequence of real numbers defined by induction by the relations

$$\begin{cases} x_1 = 1; \\ x_{n+1} = \sqrt{1 + 2x_n}. \end{cases}$$

Prove that for all $n \in \mathbb{N}$ with $n \geq 1$, we have

$$x_n < 4 \,.$$

Exercise 4. Prove the following properties of divisibility.

(a) $\forall n \in \mathbb{Z}, 1 \mid n$.

(b) $\forall d \in \mathbb{Z} \setminus \{0\}, d \mid 0$.

(c) If $d \mid n$ and $n \mid q$, then $d \mid q$.

(d) If $d \mid n$ and $d \mid q$, then $\forall u, v \in \mathbb{Z}, d \mid (un + vq)$.

(e) $d \mid 1 \Leftrightarrow d = \pm 1$.

(f) If $d \mid n$ and $n \mid d$, then $d = \pm n$.

Exercise 5. Prove that, for all $n \in \mathbb{N}$,

$$3 \mid (n^3 - n).$$

Exercise 6. Compute the quotient and remainder of the Euclidean division between the following pairs of numbers:

(a) $25, 4$.

(b) $28, 6$.

(c) $-28, 6$.

(d) $-14, 3$.

Exercise 7. Write all the elements of Div_n, the set of divisors, where n is one of the following numbers:

(a) 5.

(b) 12.

(c) 15.