10/1/2025.

## Review:

Recall: A modular form $f$ of weight $2k$ is a function $f: \mathbb{H} \to \mathbb{C}$ satisfying

$$f(z) = (cz+d)^{-2k} f\left(\frac{az+b}{cz+d}\right)$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$.

We ask for holomorphicity on $\mathbb{H}$ and at $\infty$ via

$$\tilde{f}(e^{2\pi i z}) = f(z)$$

$\to$ mero. at $z = \infty$, $q = 0$,
well-defined because of action by $T$.

If zero at $\infty$, called cusp form.

Important Example:

Eisenstein series $G_k(z) = \sum'_{m,n} \frac{1}{(mz+n)^{2k}}$ is a modular form of weight $2k$. NOT a cusp form:

$$G_k(\infty) = 2\zeta(2k)$$

We can use this to construct a cusp form:

$$g_2 \overset{\text{def}}{=} 60G_2,$$
$$g_3 \overset{\text{def}}{=} 140G_3,$$
$$\Delta = g_2^3 - 27g_3^2.$$

Easy to see weight 12, 0 at $\infty$.

We use $v_p(f)$ to denote order of $f$ at $p$ (>0 if root, <0 if pole).

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## Section 3.2:

For $k \in \mathbb{Z}$, denote by $M_k$, the $\mathbb{C}$-vector space of modular forms of weight $2k$. Let $M_k^0$ be the v.s. of cusp forms of weight $2k$. Then,

$$\operatorname{Ker}(f \mapsto f(\infty)) = M_k^0,$$

so

$$\dim(M_k / M_k^0) = \dim(\operatorname{Im}(f \mapsto f(\infty))) \leq 1.$$

(Rank-Nullity).

For $k \geq 2$, $G_k$ is a non-cusp member of $M_k$, so

$$M_k = M_k^\circ \oplus \mathbb{C} \cdot G_k \quad (k \geq 2)$$

## Theorem 4:

(i) We have $M_k = 0$ for $k < 0$, $k = 1$.

(ii) For $0 \leq k \leq 5$, $\dim M_k = 1$ with basis $1, G_2, G_3, G_4, G_5$, and $M_k^\circ = 0$.

(iii) Mult by $\Delta$ defines an iso of $M_{k-6}$ onto $M_k^\circ$.

Proof. Take $f \in M_k$, $f \neq 0$. Recall from last time

$$v_\infty(f) + \frac{1}{2} v_i(f) + \frac{1}{3} v_\rho(f) + \sum_{p \in H/G}^{*} v_p(f) = \frac{k}{6}. \quad (*)$$

All terms on LHS are $\geq 0$ (we are dealing with mod. forms), so $k \geq 0$ and $k \neq 1$ (since $\frac{1}{6}$ cannot be written in the form $n + \frac{n'}{2} + \frac{n''}{3}$ for $n, n', n'' \geq 0$. Proves (i).

Pf of (ii):

Applying $(*)$ to $f = G_k$, $k = 2$, only works if

$$0 + \frac{1}{2} \cdot 0 + \frac{1}{3} \cdot 1 + \sum 0 = \frac{2}{6}.$$

So, $v_p(G_2)=1$, $v_p(G_2)=0$ elsewhere. Similarly for $G_3$, $v_i(G_3)=1$, $0$ elsewhere.

Then,

$$\Delta = g_2^3 - 27g_3^2$$

is not identically $0$. Since weight $\Delta = 12$, $v_\infty(\Delta)\geq 1$, $(*)$ implies $v_p(\Delta)=0$ for $p\neq\infty$, $v_\infty(\Delta)=1$.

That is, $\Delta$ does not vanish on $\mathbb{H}$ and has simple zero at $\infty$.

If $f\in M_k^0$, $g:=f/\Delta$ is weight $2k-12$, and

$$v_p(g) = v_p(f)-v_p(\Delta) = \begin{cases} v_p(f) & p\neq\infty \\ v_p(f)-1 & p=\infty, \end{cases}$$

so $g\in M_{k-6}$. Similarly, if $g\in M_{k-6}$, $g\Delta\in M_k^0$. Proves (iii).

**Proof of (ii):**

If $k<6$, (iii) gives $\dim M_k^0 = \dim M_{k-6} = 0$, so $\dim M_k \leq 1$. Then, notice $1, G_2, \ldots, G_5 \in M_0, M_2, \ldots, M_5$. Proves (ii).

Corollary 1:

$$\dim M_k = \begin{cases} \lfloor \frac{k}{6} \rfloor & \text{if } k \equiv 1 \pmod 6 \\ \lfloor \frac{k}{6} \rfloor + 1 & \text{if } k \not\equiv 1 \pmod 6 \end{cases} \quad \text{for } k \geq 0.$$

Proof. Clear since $M_k = M_k^\circ \oplus \mathbb{C} G_k$, so $\dim M_k = \dim M_k^\circ + 1$.
Induct. □

Corollary 2:

The space $M_k$ has basis

all weight $k$.

$$\{ G_2^\alpha G_3^\beta : 2\alpha + 3\beta = k \}$$

Proof. Clear for $k \leq 3$, since set only has 1 elt.

For $k \geq 4$, pick one such $(\alpha, \beta)$. Take

$$g = G_2^\alpha G_3^\beta$$

(not a cusp form). For $f \in M_k$, turn into cusp form via

$$f - \lambda g.$$

By Thm 4(iii), $f - \lambda g = \Delta h$ for $h \in M_{k-6}$.

By inductive hyp,

$$h = \sum_{2\alpha+3\beta=k-6} c_{\alpha,\beta} G_2^\alpha G_3^\beta,$$

So

$$f = \lambda G_2^\alpha G_\beta^3 + \sum_{2\alpha+3\beta=k-6} (g_2^3 - 27g_3^2) c_{\alpha\beta} G_2^\alpha G_3^\beta.$$

Now, why is $\{G_2^\alpha G_3^\beta\}$ a lin. indep. set?

If

$$\sum_{\alpha,\beta} c_{\alpha,\beta} G_2^\alpha G_3^\beta = 0,$$

then divide through by some $G_2^\gamma G_3^\delta$ term.

Then, everything is weight zero:

$$\sum_{\alpha',\beta'} c_{\alpha,\beta} G_2^{\alpha'} G_3^{\beta'} = 0$$

with $2\alpha' = -3\beta'$, so this is a polynomial of 0-weight mod. form $\frac{G_3^2}{G_2^3}$. But this poly can only be 0 at finitely many values (its roots).

Since $\frac{G_3^2}{G_2^3}$ is hol., this means it is constant.

Clearly not true ($=0$ at $\rho$). $\square$

# Section 3.3:

Consider

$$j = \frac{1728 g_2^3}{\Delta},$$

weight 0.

**Proposition 5:**

(a) $j$ is a modular fn of weight 0

(b) It is hol in $\mathbb{H}$, simple pole at $\infty$.

(c) $z \mapsto j(z)$, $\mathbb{H}/G \to \mathbb{C}$ is a bijection.

Pf.

(a) clear

(b) clear $\to$ $\Delta$ is $\neq 0$ on $\mathbb{H}$, 0 at $\infty$ ($g_2^3 \neq 0$ at $\infty$).

(c)

Surjective: consider weight-12 mod form

$$f_\lambda(z) = 1728 g_2(z)^3 - \lambda \Delta(z).$$

not identically 0. By (*),

$$n + \frac{n'}{2} + \frac{n''}{3} = 1$$

for $f_\lambda$, so $f_\lambda$ can only be zero at 1 pt in $\mathbb{H}/G$. Thus, at this zero $\tau$,

$$j(\tau) = \frac{1728\, g_2(\tau)^3}{\Delta(\tau)} = \lambda.$$

Injectivity: Only one zero.

Proposition 6:

Let $f$ be a mero fn on $\mathbb{H}$. These are equivalent:

(i) $f$ is a mod fn of weight 0

(ii) $f$ is a quotient of 2 mod forms of same weight

(iii) $f$ is a rational fn of $j$.

Proof.

(iii) $\rightarrow$ (ii) $\rightarrow$ (i) Clear.

(i) $\rightarrow$ (iii): Let $f$ be a mod fn. We can get rid of pole at $\infty$ via $g = \Delta^n f$.

Since weight$(g) = 12n$, it is a lin comb of $G_2^\alpha G_3^\beta$ with $2\alpha + 3\beta = 6n$.

It is enough to show one of these satisfies (iii).

Since $2\alpha+3\beta=6n$, $p=\frac{\alpha}{2}, \frac{\beta}{3}$ are integers and so

$$f=\frac{G_2^{3p}G_3^{2q}}{\Delta^{p+q}}.$$

But $\Delta = g_2^3-27g_3^2$.

$$\frac{g_2^3}{\Delta}=\frac{j}{1728},$$

$$\frac{\Delta}{g_3^2} = \frac{g_2^3}{g_3^2}-27$$

$$=\frac{1}{27}\left(1-\frac{\Delta}{g_2^3}\right)-27\ldots$$

Done.