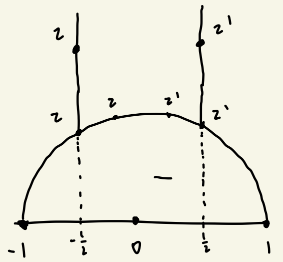
My notes 25.9.25.

Zeroes and poles of a modular function

Recall

What it means for two points $z, z' \in D$ to be congruent modulo G

$Re(z) = \pm \frac{1}{2}$ and $z = z' + 1$ or $|z| = 1$ and $z' = -\frac{1}{z}$



$a + bi = z$ s.t. $|z| = 1$

Then $-\frac{1}{a+bi} \cdot \frac{a-bi}{a-bi} = -a + bi$

f meromorphic on H, not identically zero $p \in \mathbb{H}$

Integer n s.t. $f/(z-p)^n$ is holomorphic + non-zero at p is called the order of f at p

$ord_p f = v_p(f)$

f is a modular function of weight $2k$

Identity:

$$f(z) = (cz+d)^{-2k} f\left(\frac{az+b}{cz+d}\right)$$

$v_p(f) = v_{g(p)}(f)$ if $g \in G$

$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G \subseteq SL_2(\mathbb{Z})$ c, d both can't be zero because otherwise not part of $SL_2(\mathbb{Z})$

$gz = \frac{az+b}{cz+d}$. $f(z) = 0 \longleftrightarrow f\left(\frac{az+b}{cz+d}\right) = 0$ $f(z) = \infty \longleftrightarrow f\left(\frac{az+b}{cz+d}\right) = \infty$

$v_p(f)$ depends only on image of p in $\mathbb{H}/G$ is another way to put it

Sidebar

Say f is weakly modular function

$f(z+1) = f(z)$ $f(-1/z) = z^{2k} f(z)$

Then possible to express f as a function of $q = e^{2\pi i z}$

which we call $\tilde{f}$ q series

Meromorphic on $|q| < 1$ without the center

If $\tilde{f}$ extends to meromorphic function at origin, then f is meromorphic at infinity.

$\tilde{f}$ has Laurent expansion in a neighborhood of the origin $f(q) = \sum_{-\infty}^{\infty} a_n q_n$

a_n are zero when n is small finite negative power terms

modular if this holds. $f(\infty) = \tilde{f}(0)$

Long winded way of saying we can define $v_\infty(f)$ as order of $q=0$ for $\tilde{f}(q)$

e_p - order of stabilizers of the point p

$e_p = 2$ if p is congruent modulo G to i

$e_p = 3$ if p is congruent modulo G to ρ

$e_p = 1$ otherwise

Thm

Let f be a modular function weight $2k$ not identically zero

$$v_\infty(f) + \sum_{p \in \mathbb{H}/G} \frac{1}{e_p} v_p(f) = \frac{k}{6}$$

orbits in the quotient

Another way to express:

$$v_\infty(f) + \frac{1}{2} v_i(f) + \frac{1}{3} v_\rho(f) + \sum^*_{p \in \mathbb{H}/G} v_p(f) = \frac{k}{6}$$

Σ^* denotes sum over points in $\mathbb{H}/G$ distinct from i, ρ

Intuition for finiteness:

$q = e^{2\pi i z}$ $\quad z = x+iy$

$|q| = e^{-2\pi y} \in (0,1)$ since upper Half Plane

$\tilde{f}(q) = f\left(\frac{1}{2\pi i} \log q\right)$ if $f(z) = f(z+1)$

$\tilde{f}$ meromorphic at $q=0$ $\rightarrow$ zeros / poles can only occur at $q=0$ or at isolated nonzero points

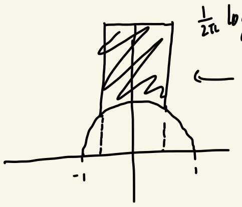
choose r very small so that $\tilde{f}$ has no zeros or poles on $0 < |q| < r$

$|q| < r \leftrightarrow e^{-2\pi y} < r \leftrightarrow y > \frac{1}{2\pi} \log \frac{1}{r}$

f has no zeros or poles when $\operatorname{Im}(z) > \frac{1}{2\pi} \log \frac{1}{r}$ except possibly at the cusp

Look at truncated domain $D_r = D \cap \{\operatorname{Im} z \le \frac{1}{2\pi} \log \frac{1}{r}\}$ This is compact

$\frac{1}{2\pi} \log \frac{1}{r}$



$\leftarrow$ compact

Zeros and poles of meromorphic function on a Riemann Surface are isolated

zeros: g holomorphic on $U \subset \mathbb{C}$

Identity theorem implies $g \equiv 0$ on component containing limit point. If multiple zeros If $g \not\equiv 0$ around p there is

Taylor Series polynomial $g(z) = (z-p)^n h(z)$ $\quad h(p) \neq 0$

we can write it like this

Poles: pole at p means $\frac{1}{g}$ is holomorphic near p and vanishes at p. Since zeros are isolated, poles are isolated

Isolated set inside a compact set gives a finite set

Thus, combining with the rest, there are only finitely many zeros and poles of f in fundamental domain

Side bar: f has zero order m at a. near a: $f(z) = (z-a)^m g(z)$ $\quad g(a) \neq 0$ holomorphic

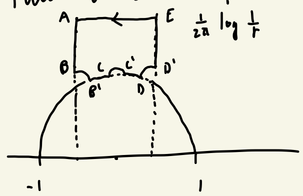
$\frac{f'}{f} = \frac{m}{z-a} + \frac{g'}{g}$ $\qquad \frac{f'}{f}$ has simple pole at a with residue m. m could be negative.

Proving Valence Formula:

Want to integrate $\frac{1}{2\pi i}\frac{df}{f}$ on boundary of D

Why? Suppose f has no zero or pole on the boundary of D except $i, \rho, -\bar{\rho}$

then we can find a contour where the interior contains representation of every pole, zero other than i, ρ



From Residue Thm: $\frac{1}{2\pi i}\int_C \frac{df}{f} = \sum_{p \in \mathbb{H}/G}^{*} v_p(f)$

Next: $q = e^{2\pi i z}$ turns EA into a circle ω center of $q = 0$ with opposite orientation

with no zeros or poles except at $q = 0$

So $\frac{1}{2\pi i}\int_E^A \frac{df}{f} = \frac{1}{2\pi i}\int_\omega \frac{df}{f} = -v_\infty(f)$

Taking integral $\frac{1}{2\pi i}\frac{df}{f}$ on circle containing arc BB' oriented negatively is $-v_\rho(f)$.

When radius $\to 0$, $\widehat{B_\rho B'} \to \frac{\pi}{3}$ so $\frac{1}{2\pi i}\int_B^{B'}\frac{df}{f} \to -\frac{1}{6}v_\rho(f)$

For CC', DD'

$\frac{1}{2\pi i}\int_C^{C'}\frac{df}{f} \to -\frac{1}{2}v_i(f)$

$\frac{1}{2\pi i}\int_D^{D'}\frac{df}{f} \to -\frac{1}{6}v_\rho(f)$

T turns AB into ED' $f(Tz) = f(z)$

$\frac{1}{2\pi i}\int_A^B \frac{df}{f} + \frac{1}{2\pi i}\int_E^{D'}\frac{df}{f} = 0$

S turns $B'C$ onto DC' $f(Sz) = z^{2k} f(z)$ $f(Sz) = z^{2k}f(z)$ $df(Sz) = 2kz^{2k-1}f(z)\,dz + z^{2k}df(z)$

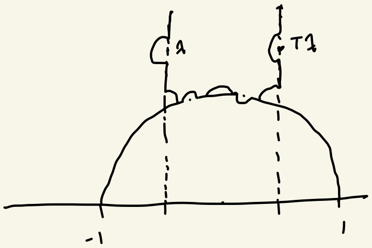
$$\frac{df(Sz)}{f(Sz)} = 2k\frac{dz}{z} + \frac{df(z)}{f(z)}$$

Since S^{-1} reverses direction

So $\frac{1}{2\pi i}\int_{B'}^{C}\frac{df}{f} + \frac{1}{2\pi i}\int_{C'}^{D}\frac{df}{f} = \frac{1}{2\pi i}\int_{B'}^{C}\frac{df(z)}{f(z)} - \frac{df(Sz)}{f(Sz)} = \frac{1}{2\pi i}\int_{B'}^{C} -2k\frac{dz}{z} = \frac{k}{6}$

What if there is a zero or pole on boundary

$z \mid Re(z) = -\frac{1}{2}$ $Im(z) > \frac{\sqrt{3}}{2}$



Going back to q-series

$f(z) = \tilde{f}(q) = \sum_{n \geq 0} a_n q^n$

If $a_0 = 0$, f is a cusp form - it vanishes at the cusp

If f is cuspform, $v_\infty(f) \ge 1$

So $\sum_{p \in \mathbb{H}/G}^{*} v_p(f) \le \frac{k}{12} - 1$

Eisenstein Series E_{2k}

For $k \ge 2$, normalized holomorphic Eisenstein Series for working with q-expansion

$E_{2k}(z) = 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n$ $\qquad E_{2k}(z) = \frac{G_{2k}(z)}{2\zeta(2k)}$

B_{2k} = 2kth Bernoulli number

$\sigma_{2k-1} = \sum_{d|n} d^{2k-1}$

What is an elliptic point?

non trivial stabilizer

If there is some element in $SL_2(\mathbb{Z})$ that stabilizes it other than 1.

For $\Gamma(1)$, i, ρ

$f(z) = (cz+d)^{-2k} f(z)$

if $(cz+d)^{-2k} \ne 1$, $f(z)=0$

if $\underbrace{(cz+d)^{-2k}}_{\text{root of unity}} = 1$, then $f(z_0)$ is nonzero

congruence tests to determine what the valence formulas are

Discriminant

$\Delta(z) = q \prod_{n=1}^{\infty} (1-q^n)^{24}$ discriminant of characteristic polynomial of $\wp'^2 = 4\wp^3 - g_2\wp - g_3$

Weight 12 cusp form

$v_\infty(\Delta) = 1$, no other interior zeros