My notes 25.9.23

Recall

Elliptic functions

Lattice functions: $\operatorname{Im}(w_1/w_2) > 0$

$\Gamma(w_1, w_2) = \mathbb{Z}w_1 \oplus \mathbb{Z}w_2 \in \mathcal{L}$

$M \to \mathcal{L}$

$$F(\lambda\Gamma) = \lambda^{-2k} F(\Gamma)$$

Eisenstein Series

$$G_k(z) = \sum_{m,n}{}' \frac{1}{(mz+n)^{2k}}$$

Def

Fundamental Parallelogram (for some Γ)

Say $\{w_1, w_2\}$ is a basis for Γ so $\Gamma = \mathbb{Z}w_1 \oplus \mathbb{Z}w_2$

then any parallelogram is one with vertices at z_0, $z_0 + w_1$, $z_0 + w_2 + w_1$, $z_0 + w_2$.

Def

Doubly Periodic Functions

Have some Lattice Γ on $\mathbb{C}$

A meromorphic function $f(z)$ on $\mathbb{C}$ is doubly periodic w/ respect to Γ

if $f(z+w) = f(z) \quad \forall w \in \Gamma$

$w = aw_1 + bw_2 \quad a, b \in \mathbb{Z}$

Prop

$f(z)$ is doubly periodic for Γ $\quad f \not\equiv 0$

D is a fundamental Parallelogram of Γ s.t. f has no poles or zeros on ∂D (If 0 on boundary, use different fundamental parallelogram)

then $\displaystyle\sum_{p \in D} \operatorname{Res}(f) = 0 \qquad \sum_{p \in D} \operatorname{ord}(f) = 0 \qquad p$ is a pole or zero

Pf sketch Residue Theorem

$$2\pi i \sum \operatorname{Res}(f) = \int_{\partial D} f = 0$$

$$\sum \operatorname{ord}(f) = \sum \operatorname{Res}\left(\frac{f'}{f}\right) = 0$$

Cor

A non constant doubly periodic function has at least two poles (by degree)

Pf

Holomorphic:

bounded in a closed parallelogram by compactness

$\hookrightarrow$ extends to $\mathbb{C}$ by periodicity

thus, constant by Liouville's Thm

One pole of degree one:

$\mathrm{Res}_p(f) = 0$ Contradiction ⋇

Simple pole has non zero residue

What about a single pole but of higher order?

Weierstrass $\wp$ - Function

We want to create some doubly periodic functions

How?

Easy when G is finite:

Say G acts on S want some function invariant under G.

Say we have $h : S \to \mathbb{C}$

Construct $f(s) = \sum_{g \in G} h(gs)$

look at $f(g's) = \sum_{g \in G} h(g'gs) = f(s)$ Every invariant function is of this form

What if not finite?

need $f(s) = \sum_{g \in G} h(gs)$ to converge (absolute) (for unconditional convergence) (failure of holomorphy / no Cauchy Integral)

for S Riemann surface, h holomorphic, need uniform convergence on compact sets for f to be holomorphic

Why: Idea from Weierstrass Thm which says that uniform convergence is needed on holomorphic

Pf comes from Cauchy's formula, need uniform to swap Σ with $\int$.

Lets say $\varphi(z)$ is a holomorphic function on $\mathbb{Z}$

$$\Phi(z) = \sum_{w \in \Gamma} \varphi(z+w)$$

Let's assume that as $|z| \to \infty$ $\varphi(z) \to 0$ fast enough that $\Phi(z)$ absolutely converges for z when $\varphi(gz)$ isn't a pole.

So $\Phi(z)$ is doubly periodic wrt Γ

Going back to

$\sum \text{ord}(f) = 0 \quad \sum \text{Res}(f) = 0$

Intuitively, simplest non constant doubly periodic function

is one with a double point at some point in Γ and nowhere else

Let us look at $f(z)$ which satisfies all of these conditions

Then $f(z) - f(-z)$ would be a doubly periodic function

with simple poles at the points in Γ if $\text{Res}_w \neq 0$. (This comes from Laurent Expansion)

We showed earlier that a non constant doubly periodic function cannot have just one pole of degree one

Thus this is constant. And by definition it is odd $(f(z) = -f(z))$ so it must be identically zero.

So f is even.

So we can do something of the sort $f(z) = \frac{1}{z^2} + O(z^2)$ near $z=0$ for normalization.

$$\frac{a_{-2}}{z^2} + \frac{a_{-1}}{z} + a_0 + a_1 z + a_2 z^2 \ldots$$

odd terms vanish.

Constant terms irrelevant because if f is valid, so is $f+c$

This is a unique form

However summing over $\frac{1}{z^2} \quad \sum_{w \in \Gamma} \frac{1}{(z-w)^2}$ this diverges

What we can do is subtract the w dependence to force convergence

$$\wp(z) = \frac{1}{z^2} + \sum_{w \in \Gamma \setminus \{0\}} \frac{1}{(z-w)^2} - \frac{1}{w^2}$$

This lacks intuition.

Better way to do it is take the derivative

$$(z^{-2})' = -2z^{-3}.$$

$$\wp'(z \mid \Gamma) = -2 \sum_{w \in \Gamma} \frac{1}{(z-w)^3}$$ which lands back to the original form.

$\wp'$ is odd which forces $\wp$ to be even

Thm $P_1 \cdots P_n$ $Q_1 \cdots Q_n \in \mathbb{C}$ repetitions allowed

but $\forall i,j$ $P_i \not\equiv Q_j \bmod \Gamma$ $P_i - Q_j \notin \Gamma$

If $\sum P_i \equiv \sum Q_j \bmod \Gamma$ then there exists a doubly periodic function $f(z)$

where its poles are P_i and zeros are Q_j and $f(z)$ is unique up to multiplication.

Pf

Def

Weierstrass Sigma function (rather complicated look it up on wikipedia)

entire, simple zeros at lattice points.

quasi periodic

$$\sigma(z+w_i) = -e^{\eta_i(z+w_i/2)}\sigma(z)$$

$$\eta_1 w_2 - \eta_2 w_1 = 2\pi i$$

$$f(z) = e^{az}\frac{\prod_{j=1}^{n}\sigma(z-Q_j)}{\prod_{i=1}^{n}\sigma(z-P_i)}$$

$$\sum P_i - \sum Q_j \overset{\in\Gamma}{=} m_1 w_1 + m_2 w_2$$

$$a = -m_1\eta_1 - m_2\eta_2$$

$$\frac{f(z+w_i)}{f(z)} = \exp\left(a w_i + \eta_i\left(\sum P_i - \sum Q_j\right)\right)$$

$$a w_i + \eta_i\left(\sum P_i - \sum Q_j\right) = -m_1\eta_1 w_i - m_2\eta_2 w_i + m_1\eta_w w_1 + m_2\eta_w w_2 = m_1(\eta_i w_1 - \eta_1 w_i) + m_2(\eta_i w_2 - \eta_2 w_i)$$

Multiple of $2\pi i$.

So $f(z+w_i) = f(z)$.

Another Proof via Riemann Roch

Side tangent:

Quotient of C by lattices

Γ is a lattice in C

Then we can make C/Γ a Riemann space

$\overset{\in V}{Q} \in C$ $\overset{\in U}{P}$ is its image in C/Γ

$p: C \to C/\Gamma$

$(U, p^{-1}(U))$

$L: \mathbb{R}^2 \to \mathbb{C}$ $\quad L(x,y) = x\omega_1 + y\omega_2$

$L(\mathbb{Z}^2) = \Gamma$ so

$$\mathbb{R}^2/\mathbb{Z}^2 \cong \mathbb{C}/\Gamma$$

and $\mathbb{R}^2/\mathbb{Z}^2 \cong S^1 \times S^1$

Why care?

$\mathbb{C}/\Gamma$ is a Riemann surface genus 1

(mod 2)

$z \to (\wp(z), \wp'(z))$

creates a biholomorphism

from $\mathbb{C}/\Gamma$ to field of doubly periodic functions $\mathbb{C}(\wp(z), \wp'(z)) \cong \mathbb{C}(x,y)/y^2 - 4x^3 - g_2 x + g_3$

Eisensteins