Mod Forms Notes

We've seen the fundamental domain of mod-forms (ie where they live)
Proved some basic ideas about them

Q: Do they exist?

A: Yes (wouldn't be interesting seminar if not!)

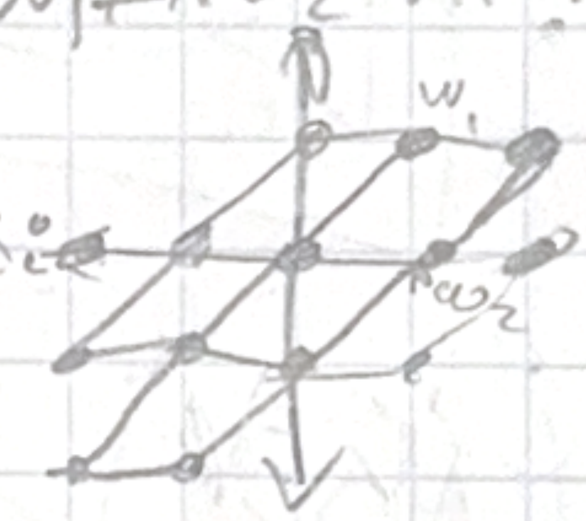
Q: How do we explicitly construct?

A: We'll define lattice fns, show how (some) lattice fns are related to mod fns, and then construct our first examples of mod forms (including a cusp form)

§ Modular Functions + Lattice Functions

(present way) Def: A lattice over $\mathbb{C}$ is $\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$, for $\omega_1, \omega_2 \in \mathbb{C}$, $\omega_1 \neq \lambda\omega_2 \ \forall \lambda \in \mathbb{R}$.

Ex:



Def: A modular function of weight $2k$ is $f: \mathbb{H} \to \mathbb{C}$ s.t.
f merom on $\mathbb{H}$, $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$

Note (w/out pf, see Serre) f merom on $\mathbb{H} \Rightarrow f$ weakly mod $\iff f(z+1) = f(z)$
$f\left(-\frac{1}{z}\right) = z^{2k} f(z)$
Obviously uses $S + T$ generating $PSL_2(\mathbb{Z})$

Rmk f mod form $\Rightarrow f$ mod fn, so these mod fns are a "natural" place to look for our first mod forms.

Back to lattices!

Let $M = (\omega_1, \omega_2) \in \mathbb{C}^*$ s.t. $\operatorname{Im}(\omega_1/\omega_2) > 0$. Define $\Gamma(\omega_1, \omega_2) := \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$

This is clearly a surjective map onto the space of lattices of $\mathbb{C}$. Clearly, map not injective, which pairs produce same lattice?

Idea $g \in SL_2(\mathbb{Z}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $(\omega_1, \omega_2) \in M$. Write $\omega_1' = a\omega_1 + b\omega_2$ $\omega_2' = c\omega_1 + d\omega_1$. Because our det is one $\Rightarrow$ this is inv $\Rightarrow \{\omega_1', \omega_2'\}$ still basis! Also, writing $z = \omega_1/\omega_2$ $z' = \omega_1'/\omega_2'$, and $gz = z' \Rightarrow \operatorname{Im}(z') > 0$, since $PSL_2(\mathbb{Z})$ maps from $\mathbb{H} \to \mathbb{H}$.

Claim Letting $SL_2(\mathbb{Z})$ act in this way actually classifies which pairs generate the same lattice

Pf Half already done, see Serre for details

Modding out by $\mathbb{C}^*$ acting how you'd expect $T(\omega_1, \omega_2) \to T(\lambda\omega_1, \lambda\omega_2)$, we see $M/\mathbb{C}^* = \mathbb{H}$ (ie send $(\omega_1, \omega_2) \longrightarrow \omega_1/\omega_2$) and the $SL_2(\mathbb{Z})$ acting now acts like $PSL_2(\mathbb{Z})$)

Rmk1: In some sense lattice space $\cong$ space where mod forms live.

Rmk2: If you're interested in elliptic curves, one can also notice this the isomorphism class of elliptic curves.

§ Lattice Functions

Def: A lattice function is a complex-valued function on the space of lattices ($= M/SL_2(\mathbb{Z})$). We say that F a lattice is of weight $2k$ if $F(\lambda T) = \lambda^{-2k} F(T)$ $\forall$ T lattice, $\lambda \in \mathbb{C}^*$

Letting (by abuse of notation) F be a function on M via $F(\omega_1, \omega_2) = F(T(\omega_1, \omega_2))$, we see $F(\lambda\omega_1, \lambda\omega_2) = \lambda^{-2k} F(\omega_1, \omega_2)$.

Because $T(\omega_1, \omega_2) = T'(\omega_1', \omega_2') \iff g(\omega_1, \omega_2) = \omega_1', \omega_2'$ (as def'd earlier), we see this function is invariant under $SL_2(\mathbb{Z})$.

We now notice that $(\omega_2)^{2k} F(\omega_1, \omega_2)$ is well defined up to $z = \frac{\omega_1}{\omega_2}$, since $(\lambda\omega_2)^{2k} F(\lambda\omega_1, \lambda\omega_2) = \lambda^{2k}\omega_2^{2k}\lambda^{-2k} F(\omega_1, \omega_2) = \omega_2^{2k} F(\omega_1, \omega_2)$. Thus, if we define $f(\underset{=\omega_1/\omega_2}{z}) = \omega_2^{2k} F(\omega_1, \omega_2)$, we get an honest function on $\mathbb{H}$.

Since F invariant under action of $SL_2(\mathbb{Z})$, we see that

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^{2k} F(az+b, cz+d) = (cz+d)^{2k} F(z,1) = (cz+d)^{2k} f(z)$$

$\exists$ if f merom, this f is a modular function of wt $2k$!

Further, writing $F(\omega_1, \omega_2) = \omega_2^{-2k} f(\omega_1/\omega_2)$ if f is a modular fn

We can get back a lattice function. This sets up a correspondence

$$\left\{\begin{matrix}\text{some Lattice fns}\\ \text{of weight}\\ 2k\end{matrix}\right\} \Longleftrightarrow \left\{\begin{matrix}\text{mod fns}\\ \text{of weight}\\ 2k\end{matrix}\right\}$$

note that not all Lattice fns produce mod fns

Examples of Mod Functions (Forms)

Lemma: If Γ lattice, then $\sum_{\gamma \in \Gamma}' \frac{1}{|\gamma|^{\sigma}}$ converges for $\sigma > 2$, $\sum'$ means sum over non-zero elements.

Pf Consider $\iint \frac{dx\,dy}{(x^2+y^2)^{\sigma/2}}$ integrated over the plane $\setminus$ some small disk abt 0.

Now, let $k>1$. Then write

Def $G_{2k}(\Gamma) = \sum_{\gamma \in \Gamma}' \frac{1}{\gamma^{2k}}$. By the last lemma, this converges absolutely.

Def: $G_{2k}(\omega_1, \omega_2) = \sum_{m,n}' \frac{1}{(m\omega_1 + n\omega_2)^{2k}}$ ← as before, lattice fn gives us fn on M ($\in M$)

Def: $G_{2k}(z) = \sum_{m,n}' \frac{1}{(mz+n)^{2k}}$ ($z \in \mathbb{H}$) ← as earlier, choosing $\omega_1 = z, \omega_2 = 1 \Rightarrow$ get mod fn on $\mathbb{H}$

Claim: $G_{2k}(z)$ is a mod form of weight $2k$. We call G_{2k} the eisenstein series of index $2k$

Pf: Construction from earlier tells us that this satisfies functional eq. of wt $2k$. We show it's holomo. For $z \in D$, $|mz+n|^2 \geq |m\rho - n|^2$, $\rho = e^{\frac{2\pi i}{3}}$, which converges $\Rightarrow$ normal cvgence on D (and on gD) $\Rightarrow$ holom on $\mathbb{H}$ (normal limit of holom fns on sets covering $\mathbb{H}$, since gD covers $\mathbb{H}$)

to see holom at inf, we send $z \to i\infty$, we see that all terms w/ $m \neq 0$ vanish, leaving $G_{2k}(\infty) = \sum_{n \in \mathbb{Z}}' \frac{1}{(n)^{2k}} = 2\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = 2\zeta(2k)$

$\Rightarrow$ We have an infinite family of mod forms G_{2k}

§ Elliptic Curves (this mostly an aside, proving ∃ of cusp form using Ell curves) do sums

We notice:
- Can multiply mod forms by constants
- Can add mod forms of the same weight
- Can mult mod forms by other mod forms

Define (for convenience): $g_2 = 60G_4$ ← wt 4, $g_3 = 140G_6$ ← wt 6

We take notation G_n, for n even, is mod form of wt n. Often, people write g_k to be wt $2k$, so I'm adopting this notation just for this part, to match references

Then $g_2(\infty) = \frac{4}{3}\pi^4$ $g_3(\infty) = \frac{8}{27}\pi^6$

Then: One can check that $\Delta := g_2^3 - 27g_3^2$ is a mod form of weight 12, w/ $\Delta(\infty) = 0$ ← recall that we call Δ a "cusp form"

Q: Is $\Delta \equiv 0$?

Def For a given lattice T of $\mathbb{C}$, define the Weierstrass $\wp$-function

meromorphic w/ poles on T → $\wp_T(u) = \frac{1}{u^2} + \sum_{\gamma \in T}' \left(\frac{1}{(u-\gamma)^2} - \frac{1}{\gamma^2}\right)$. Up to taking a Laurent Expansion, we can see $\wp(u) = \frac{1}{u^2} + \sum_{k=1}^{\infty} (2k-1) G_k(T) u^{2k-2}$.

Then, taking derivatives and examining coefficients, we can see that $\forall u, T$: $(\wp_T'(u))^2 = 4\wp_T(u)^3 - g_2(T)\wp(u) - g_3(T)$ (see Milne Ch3 for pf or maybe? next presentation)

One can prove this is ≅ to elliptic curve $\mathbb{C}/T$

$\Rightarrow$ non singular $\Rightarrow$ discrim $\neq 0 \Rightarrow g_2^3 - 27g_3^2 \neq 0 \Rightarrow \Delta \neq 0$, so its a non-trivial cusp form

Note: All of this non-obvious, needs proof (which I won't provide, I think it's proved in Silverman's Arithmetic of Ell Curves, or Milne) — Just pointing out connection