My notes 25.9.17.

<u>Recall</u>

$\Gamma(1) = SL_2(\mathbb{Z})$

Fundamental Domain connected open subset

$D \subset \mathbb{H}$ where no two points are equivalent under $\Gamma$

$\Gamma \underset{\text{discrete}}{\subseteq} SL_2(\mathbb{R})$

stabilizers are not $\pm I_2$ : $i, \rho, \rho^2$

$z \in \mathbb{H}$ elliptic point if it is the fixed point of elliptic $\gamma \in \Gamma$

$s \in \mathbb{R} \cup \{\infty\}$ cusp if $\exists \gamma \in \Gamma$ parabolic with $s$ as the fixed point

<u>Today</u>

Look at $\mathbb{H}^*$ and structure on $\Gamma \backslash \mathbb{H}^*$

<u>ex</u>

<u>Lemma</u>

Automorphisms of $D$ fixing $0$ are of the form $z \mapsto \lambda z$ with $|\lambda| = 1$

<u>Pf</u>

Comes from something called Schwarz Lemma: $f$ holomorphic on $|z|<1$ $f(0)=0$, $f(z)<1$ when $|z|<1$ then $|f| \leq |z|$ when $|z|<1$

$D$ open unit disk $|z|<1$

$\Delta$ finite group acting on $D$ fixing $0$

$\mathrm{Aut}(D, 0) = \{ z \in \mathbb{C} \mid |z| = 1 \} \cong \mathbb{R}/\mathbb{Z} \to$ finite cyclic group

let $\zeta$ be the generator with $\zeta^m = 1$.

$z^m$ is invariant on $D$ and so it defines a function on $\Delta \backslash D$

this is continuous and open on $D \setminus \{0\}$

Homeomorphism $\Delta \backslash D \to D$ and so $z^m$ defines a complex structure on $D$

quotient map is proper and closed.

$p$ quotient map $D \to \Delta \backslash D$

$f \mapsto f \circ p$ bijection from holomorphic functions on $U \subset \Delta \backslash D$ to holomorphic

So we have uniqueness

Structure on $\Gamma(1) \backslash \mathbb{H}^*$

$p$ quotient map $\mathbb{H} \to \Gamma(1) / \mathbb{H}$

$Q$ is a point in $\mathbb{H}$ which maps to $P$ a point in $\Gamma(1) \backslash \mathbb{H}$

If $Q$ isn't elliptic point: it is trivial

nbhd $U \ni Q$ s.t. $U \to p(U)$ homeomorphism, then define $(p(U), p^{-1})$ to be coordinate nbhd of $P$

inverse gives usual holomorphic coordinate on $\varphi$ near $P = p(Q)$

What if it is an elliptic point:

<u>Look at $Q = i$</u>

then we can just take it at $i$. need a small Disc that is $S$ stable $S$ maps to itself

$\gamma \in \Gamma(1)$, $\gamma = S(\,)$

$S(D) = D$ $\quad \gamma(D) \cap D = \emptyset$

$z \longmapsto \dfrac{z-i}{z+i}$ isomorphism from open nbhd of $D$ of $i$ stable under $S$ Recall $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ it's the one that flips it.

Cayley map

goes to an open disk $D'$ with center $0$ and action of $S$ on $D$ is $\sigma : z \to -z$ on $D'$. $Sz = -z$

$\langle S \rangle \backslash D$ is homeomorphic to $\langle \sigma \rangle \backslash D'$ and so $\langle S \rangle \backslash D$ gets the complex structure and so we have

bi-holomorphic isomorphism. $\langle \sigma \rangle \backslash D'$

$\dfrac{z-i}{z+i}$ holomorphic. $S$ turns it into $-\dfrac{z-i}{z+i}$. So quotient by $180^\circ$ rotation

so $\left(\dfrac{z-i}{z+i}\right)^2$ is a holomorphic fct defined in nbhd of $i$ invariant under the action of $S$.

So it defines a holomorphic function in nbhd of $p(i)$ so it is the coordinate function near $p(i)$

Look at $Q = \rho^2$

$\rho^2$ fixed by $ST$. $\dfrac{z-\rho^2}{z-\bar{\rho}^2}$ maps to disk center $0$

$\left(\dfrac{z-\rho^2}{z-\bar{\rho}^2}\right)^3$ is invariant under $ST$

So this defines a nbhd of $p(\rho^2)$ and this is the coordinate function near $p(\rho^2)$

There is an issue: $\Gamma(1) \backslash \mathbb{H}$ is not compact

need to add the point $\infty$ to $\mathbb{H}$

also need to map this point

$q(z) = \exp(2\pi i z)$ to some neighborhood $U$ of $\infty$

$q$ is invariant under action of stabilizer of $\langle T \rangle$

So $q : \langle T \rangle \backslash U \to U$ is a holomorphic function which we take to be the coordinate fct. near $p(\infty)$

Another method:

$H^* = \mathbb{H} \cup P^1(\mathbb{Q})$

set of cusps for $\Gamma(1)$.

Every cusp other than $\infty$ is rational point on real axis

and is of the form $\sigma\infty$ $\sigma \in \Gamma(1)$. $\sigma\infty$ gets the fundamental system of nbhds for which $\sigma$ is a homeomorphism. So $\Gamma(1)$ is continuous on $\mathbb{H}^*$.

so we have $\Gamma(1)\backslash\mathbb{H}^* = (\Gamma(1)\backslash\mathbb{H}) \cup \{\infty\}$

<u>Prop</u> $\Gamma(1)\backslash\mathbb{H}^*$ compact

<u>pf</u> $\overline{D} \cup \{\infty\}$ is compact

<u>Prop</u> $\Gamma(1)\backslash\mathbb{H}^*$ genus zero

<u>Pf sketch</u>

It is simply connected.

<u>$\Gamma(1)\backslash\mathbb{H}^*$ isomorphic to Riemann Sphere</u>

Only simply connected compact Riemann Surface

<u>Structure on $\Gamma\backslash\mathbb{H}^*$</u>

$\Gamma \subset \Gamma(1)$ of finite index.

We can define $\Gamma\backslash\mathbb{H}^*$ in similar way.

Complement of $\Gamma\backslash\mathbb{H}$ in $\Gamma\backslash\mathbb{H}^*$ is set of equiv. classes of cusps for $\Gamma$

$\Gamma\backslash\mathbb{H}$ given complex structure in similar way as $\Gamma(1)$.

$\infty$ always a cusp.

$h$ is smallest power of $T$ in $\Gamma$

$q = \exp(2\pi i z/h)$ is coordinate fct. near $\infty$.

Any other cusp is of the form $\sigma\infty$ for $\sigma \in \Gamma(1)$

with $z \longmapsto q(\sigma^{-1}(z))$ being the coordinate function near $\sigma\infty$.