① Riemann Surfaces
holomorphic functions
meromorphic functions
sheaves

② Analysis on R.S.
differential forms / integration
Residue thm
R-R thm

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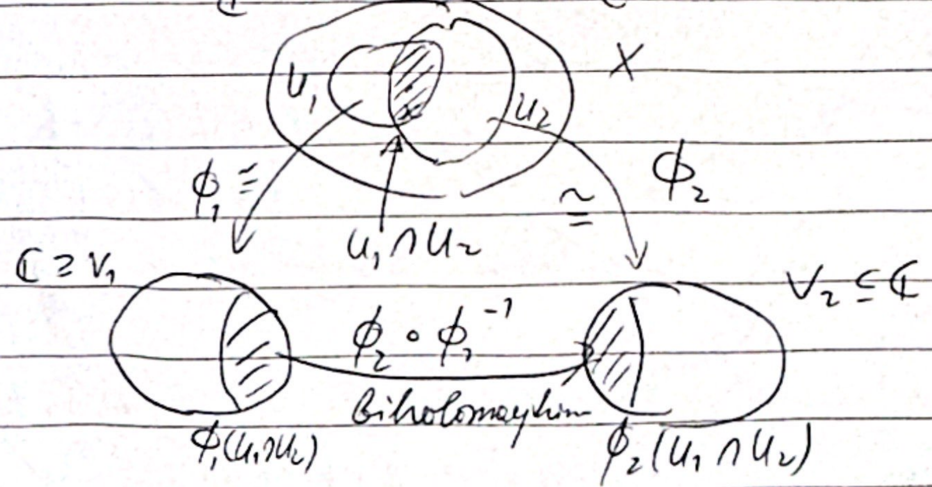
Def: 1) A topological space X is a set X with a topology $\mathcal{T} \subseteq \{\text{subsets of } X\}$ s.t. ① $\phi, X \in \mathcal{T}$ ② $\{U_\alpha\} \subseteq \mathcal{T} \Rightarrow \bigcup U_\alpha \in \mathcal{T}$
③ $\{U_1, \cdots, U_n\} \subseteq \mathcal{T} \Rightarrow \bigcap_{i=1}^{n} U_i \in \mathcal{T}$

2) Given X, Y top. space. $f: X \to Y$ is a homeomorphism if ① f is a bijection ② $f(U) \subseteq Y$ is open $\Leftrightarrow U \subseteq X$ open

3) $f: X \to Y$ is a biholomorphism if f ~~area~~ homeo + holomorphic & f^{-1}

Def: A Riemann Surface X is a topological space s.t.
①. $\forall x \in X$, $\exists$ nbhd $U \subseteq X$, $V \subseteq \mathbb{C}$ open, and a homeo $\phi: U \xrightarrow{\sim} V$
② If $\phi_1: U_1 \xrightarrow{\sim} V_1$, $\phi_2: U_2 \xrightarrow{\sim} V_2$ local charts then
$\phi_2 \circ \phi_1^{-1}: \phi_1(U_1 \cap U_2) \xrightarrow{\sim} \phi_2(U_1 \cap U_2)$ is a biholomorphism.
$\bigcap\!\!\!|\; \mathbb{C} \qquad\qquad \bigcap\!\!\!|\; \mathbb{C}$



E.g. 1) $\mathbb{C}$ 2) $\mathbb{CP}^1 = \{(Z, W) \in \mathbb{C}^2 - \{0\}\} / \mathbb{C}^* = \{[Z:W]\}$
Two open nbhd: $U_1 = \mathbb{CP}^1 \setminus [0:1] = \{[Z:W] \in \mathbb{CP}^1 \mid Z \neq 0\}$ &
$U_2 = \mathbb{CP}^1 \setminus [1:0] = \{[Z:W] \in \mathbb{CP}^1 \mid W \neq 0\}$
and homeom $\phi_1: U_1 \xrightarrow{\sim} \mathbb{C} \quad [Z:W] \mapsto \frac{W}{Z}$
$\phi_2: U_2 \xrightarrow{\sim} \mathbb{C} \quad [Z:W] \to \frac{Z}{W}$
On $U_1 \cap U_2$, $\mathbb{C} - \{0\} \xrightarrow{\phi_1^{-1}} \mathbb{CP}^1 - \{[1:0], [0:1]\} \xrightarrow{\phi_2} \mathbb{C} - \{0\}$
$t \longmapsto [1:t] \longmapsto \frac{1}{t}$

3) Complex torus: Fix $\omega_1, \omega_2 \in \mathbb{C}^*$ $\mathcal{L} := \{n_1\omega_1 + n_2\omega_2 \mid n_1, n_2 \in \mathbb{Z}\} \subseteq \mathbb{C}$
$\mathbb{Z}$-linear independent

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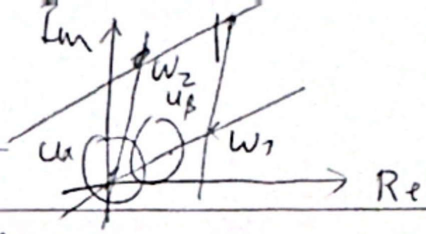
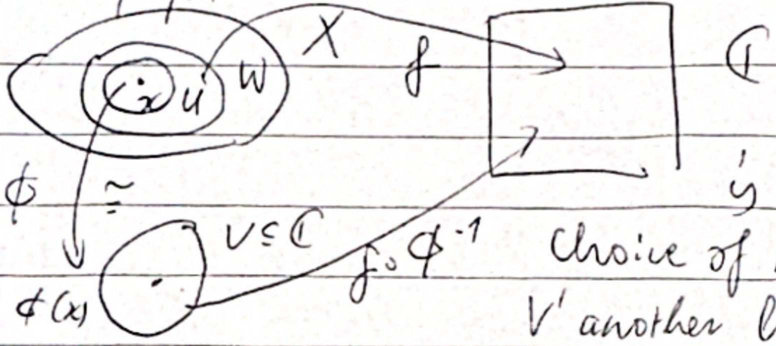


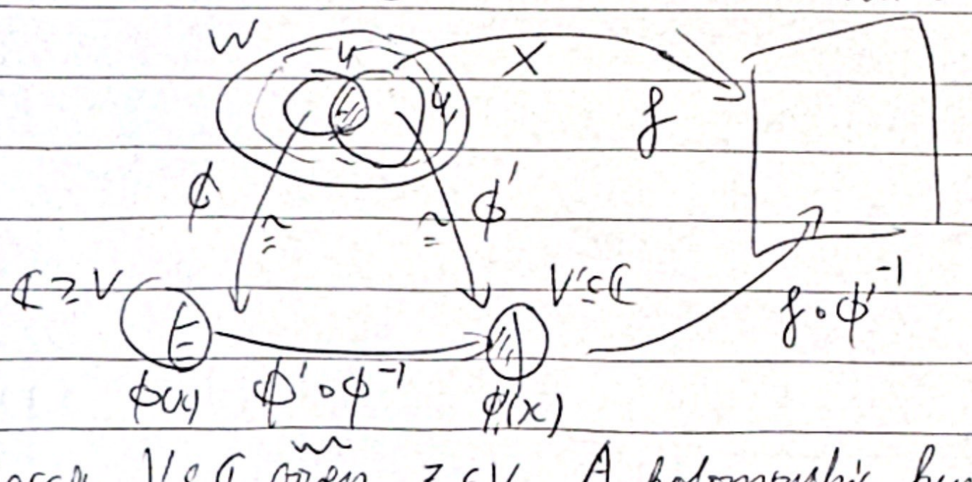
Equivalence relation on $\mathbb{C}$: $z \sim z' \iff z' - z \in \mathcal{L}$. Let $X = \mathbb{C}/\sim$ with quotient topology. $\phi_\alpha : U_\alpha \xrightarrow{\sim} U_\alpha$ $\phi_\beta : U_\beta \to U_\beta$ ...

X: R.S.

Def: Say $W \subseteq X$ open. A function $f : W \to \mathbb{C}$ is holomorphic at $p \in W \subseteq X$ if $\exists$ local chart $\phi : \underset{p \in}{U} \xrightarrow{\sim} V$, $U \subseteq W$ s.t. $f \circ \phi^{-1} : \underset{\mathbb{C}}{V} \to \mathbb{C}$ is holomorphic at $\phi(p)$.



Note that this is independent of choice of nbhd: if $\phi' : U' \xrightarrow{\sim} V'$ another local chart $f \circ \phi'^{-1}$ is also holomorphic at $\phi'(p)$ since $f \circ \phi'^{-1} = (f \circ \phi^{-1}) \circ (\phi \circ \phi'^{-1})$ $= f \circ \phi^{-1} \circ$ (Biholomorphic)



Recall: $V \subseteq \mathbb{C}$ open. $z_0 \in V$. A holomorphic function $f : V \setminus \{z_0\} \to \mathbb{C}$ defined on a punctured z_0 can have 1 of following 3 limiting behaviors around z_0 ① Removeable singularity.

② Pole: $f = \frac{1}{(z-z_0)^n} \tilde{f}$ for some $\tilde{f} : V \to \mathbb{C}$, $n < +\infty$ (Laurent series expansion)

③ Essential singularity (infinite many $n<0$ $f = \sum_{-\infty}^{\infty} a_n (z-z_0)^n$)

If $f : V \setminus \{z_0\} \to \mathbb{C}$ is ① or ②, then say f is meromorphic at z_0.

Def: X: R.S. Say $W \subseteq X$ open, $p \in W$. A holomorphic function $f : W \setminus p \to \mathbb{C}$ is a meromorphic at p if $f \circ \phi^{-1} : V \setminus \phi(p) \to \mathbb{C}$ has removeable/pole at $\phi(p)$

- The order of meromorphic function at a point

Def: Let $f: W \dashrightarrow \mathbb{C}$ be a meromorphic function and $p \in W$. Fix one local chart $\phi: U \to V$ around p and say $\phi(p) = z_0$. If $(f \circ \phi^{-1})(z) = \sum_{i \geq r} c_i (z-z_0)^i$ is the Laurant series around z_0, then define $r = \min\{i \mid c_i \neq 0\}$ to be the order of f at p

Prop: X: Cpt. R.S. Then every holomorphic function $f: X \to \mathbb{C}$ is constant

proof sketch: $f(X) \subseteq \mathbb{C}$ cpt. subspace i.e. closed & bounded. $\exists$ a point $p_0 \in X$ st. $|f(p_0)|$ is maximum among $|f(X)|$. Let $\phi: U \to V$ be a local chart around p, then $f \circ \phi^{-1}: V \to \mathbb{C}$ attains max value. By max modulus thm $\Rightarrow$ $f \circ \phi^{-1}$ constant. For each $q \in X$, connect p, q by a line & cover it by a finite # of local charts. By identity thm, f is constant $\square$

Prop: X: Cpt. R.S. $f: X \dashrightarrow \mathbb{C}$ meromorphic function then $\exists$ finitely many zero & poles

proof: {zeros & poles} discrete $\subseteq X \Rightarrow$ finite $\square$

Thm: Every meromorphic function on $\mathbb{CP}^1$ is of the form rational $f([x:y]) = \prod_{i=1}^{N} (a_i x - b_i y)^{e_i}$, $e_i \in \mathbb{Z}$ (with $\sum_{i=1}^{N} e_i = 0$)

proof: If $g(z)$ meromorphic, then $< \infty$ many zeros & poles & may assume no zero nor pole at $(\infty = [0:1])$. Say $g(z)$ has poles of order m_i at p_i, zeros of order n_i at q_i. then $g(z) \dfrac{\prod (z-p_i)^{m_i}}{\prod (z-q_i)^{n_i}}$ has no pole nor zero, hence constant. So $g(z) = \dfrac{\prod (z-q_i)^{n_i}}{\prod (z-p_i)^{m_i}}$ $\square$

Sheaves: (of Ab.)

Def: A presheaf is a functor $F: \mathrm{Ouv}^{op} \to \mathrm{Ab}$

where Ouv is the category with obj = open sets of X and $\mathrm{Mor}(U, V) = \begin{cases} \phi & \text{else} \\ \{*\} & \text{if } U \hookrightarrow V \end{cases}$

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A presheaf $\mathcal{F}$ is a sheaf if $\forall U \subseteq X$. $U = \bigcup_\alpha U_\alpha$

1). if $s, t \in \mathcal{F}(U)$ s.t. $s|_{U_\alpha} = t|_{U_\alpha}$ for all $\alpha \in I$, then $s = t \in \mathcal{F}(U)$

2). If $s_\alpha \in \mathcal{F}(U_\alpha)$ a collection of sections agree on intersections. then $\exists\ s \in \mathcal{F}(U)$ s.t. $s|_{U_\alpha} = s_\alpha$.

E.g. $\mathcal{O}_X$ sheaf of holomorphic functions, $\mathcal{M}_X$ ... meromorphic functions — sheaf of $\mathbb{C}$-algebra. Ω_X cotangent sheaf ... sheaf of $\mathcal{O}_X$-module.

$(X, \mathcal{O}_X)$ ringed space. Morphism of ringed space $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ $f: X \to Y$ & $f_1\ \mathcal{O}_Y \to f_*\mathcal{O}_X$: ($\forall U' \subseteq Y$, $g \in \mathcal{O}_Y(U') \to g \circ f \in \mathcal{O}_X(f^{-1}(U'))$)

Analysis on R.S.

Differential forms (Integration.

Recall: $V \subseteq \mathbb{C}$ open. a holomorphic 1-form ω on V is a formal symbol $\omega = f(z)dz$ where $f(z)$ holomorphic

a meromorphic 1-form ω on V ...

$\omega = g(z)dz$ where $g(z)$ meromorphic

Def: X: R.S. A holomorphic 1-form ω on X is a collection of holomorphic 1-form ω_i on V_i, where $\phi_i: U_i \xrightarrow{\sim} V_i$ local chart. s.t. if ϕ_i, ϕ_j 2 local charts with transition function $T: V_i \to V_j$, then $f_i(z) = f_j((T(z))\, T'(z)$.

A merom. 1-form ω ...

Def: ω meromorphic 1-form. $\mathrm{Ord}_p\omega := \mathrm{ord}_{\phi(p)} f(z)$.

Residue:

Recall: $f: \mathbb{C} \to \mathbb{C}$ holomorphic, γ closed path. then $\oint_\gamma f\,dz = 0$

Thm. f meromorphic. $\int_\gamma f\,dz = 2\pi i \cdot C_{-1}$. where $f(z) = \sum_{n \geq -r} C_n (z - z_0)^n$ Laurent series. C_{-1} the coeff of z^{-1} term

Def: $\mathrm{Res}_{z_0} f = C_{-1}$

Def: X: R.S. ω: meromorphic 1 form on X and $p \in X$.
Define $\mathrm{Res}_p \omega$ as follows: take a local coordinate z around $p \in X$. then $\omega = \sum_{n \geq -r} c_n z^n dz$. Take $\mathrm{Res}_p \omega = \mathrm{Res}_{\phi(p)} f$ $\phi: U \xrightarrow{\sim} V \subseteq \mathbb{C}$

Lemma: (Residue Thm) $\mathrm{Res}_p \omega = \frac{1}{2\pi i} \int_\gamma \omega$ for any small loop γ around $p \in X$

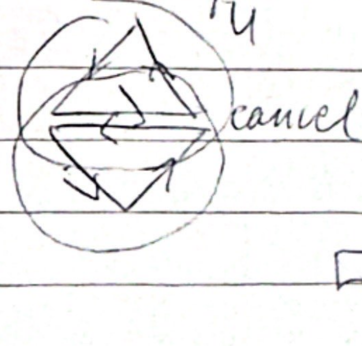
proof: Take local chart $\phi: U \xrightarrow{\sim} V \Rightarrow \omega = f(z) dz = \sum_{n \geq -r} c_n (z - z_0) dz$ for merom function $f(z)$ on $V \subseteq \mathbb{C}$. Then $c_{-1} = \frac{1}{2\pi i} \int_\gamma f(z) dz$ □

Note. $\int_\gamma \omega = \sum_{i=1}^{r} \int_{a_i}^{b_i} f_i(z) dz$. Take local expression $\omega_i = f_i dz$

$$\begin{tikzcd} {[a_i, b_i]} \arrow[r, "\gamma_i"] \arrow[dr, "z"'] & U_i \arrow[d, "\cong"] \\ & V_i \subseteq \mathbb{C} \end{tikzcd}$$

Thm: (Residue Thm for cpt R.S.). X cpt R.S. ω merom 1-form. Then $\sum_{p \in X} \mathrm{Res}_p \omega = 0$

proof sketch: Fact: X: cpt R.S. Triangulizable So we may choose the triangulization s.t. each pole of ω lies in the interval of one of the triangles. And choose a positive orientation on the boundary of each triangle.



Now $X = \Delta_1 \cup \cdots \cup \Delta_N$.
So $0 = \sum_{j=1}^{N} \int_{\partial \Delta_j} \omega = \begin{cases} 0 & \text{if } \Delta_j \cap \{\text{poles}\} = \phi \\ \left(\sum_{p \in \Delta_j \cap \{\text{poles}\}} \mathrm{Res}_p \omega \right) 2\pi i \end{cases}$ □

Divisors & Riemann-Roch

Def: (1) A divisor on X is a formal $\mathbb{Z}$ linear combination of points in X: $D = \sum_{p \in X} a_p \cdot p$ where $\mathrm{Supp}\, D = \{p \in X \mid a_p \neq 0\}$ is discrete
(2) $\mathrm{Div}(X) := \{\text{all divisors on } X\}$ ab. gp.
(3) D is effective if $a_p \geq 0$ $\forall p$.
(4) f meromorphic. $\mathrm{div} f := \sum \mathrm{ord}_p(f) \cdot p$. principal divisors.

(5) $D \sim D'$ if $D - D' = \underset{\sum a_p \cdot P}{\overset{\shortparallel}{\mathrm{div}\, g}}$ for g merom.

(6) $\deg : \mathrm{Div}(X) \to \mathbb{Z}$ $D \mapsto \sum a_p$.

(7) $\mathrm{div}\, \omega = \sum \mathrm{ord}_p \omega \cdot P$ Canonical divisor $=: K$.

Remark: $\mathrm{div}(fg) = \mathrm{div} f + \mathrm{div}\, g$; $\mathrm{div}(1/f) = -\mathrm{div} f$

$\mathrm{div}(f\omega) = \mathrm{div} f + \mathrm{div}\, \omega$

Def: $\mathcal{L}(D) = \{ f \in \mathcal{M}(X) : \mathrm{div} f + D \geq 0 \}$

$H^0(X, D) = \ell(D) = \dim_{\mathbb{C}} \mathcal{L}(D) =$

Thm (Riemann-Roch). X cpt. R.S. $\exists\, g \geq 0$ $g \in \mathbb{Z}$. $\forall D$.

$\ell(D) = \deg(D) + 1 - g - \ell(K - D)$

g is the genus of X

Thm (Riemann-Hurwitz).