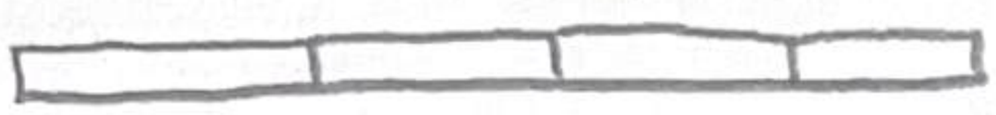
12/03/2025: Sphere Packing 1

Question: What is the densest configuration of non-overlapping equally sized balls in the d-dimensional Euclideans?

Intuition: Very big box, infinite balls. How many balls fit? If box is much bigger than balls, look at density (i.e $V(box) \cdot \delta$ constant). Balls are rigid and nonmalleable, Can touch but not intersect.

Let's look at 2 easy examples for $d=1$ and $d=2$.

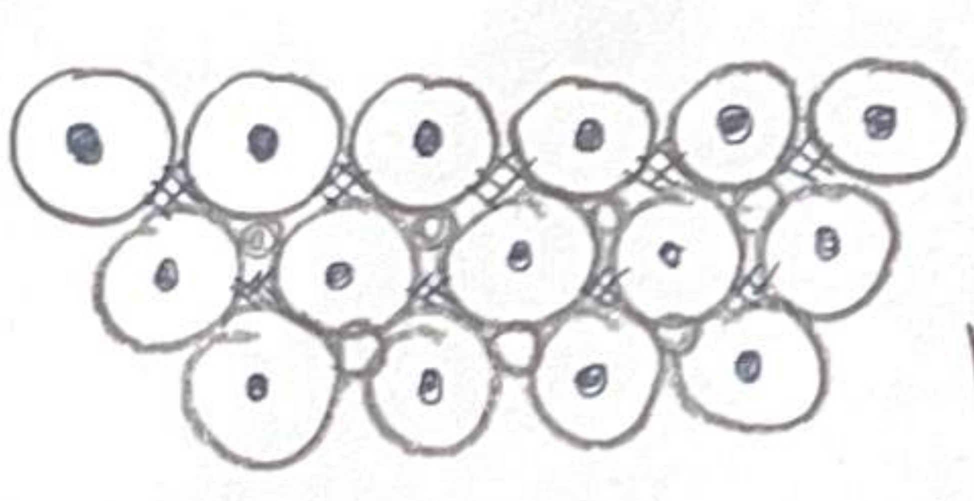
$n=1$:  Density $= 1$. Only 1 degree of freedom. Trivially covered by intervals perfectly, and density $= 1$.

$n=2$: 

Idea: 2 degrees of freedom (NE + SW). How best to cover the surface of table with coins.
"Honeycomb" Lattice $\Rightarrow$ Density $= \frac{\pi}{\sqrt{12}} \approx 0.9069$

$n=3$: Kepler Conjecture Density $\pi/\sqrt{18} \approx 0.7405$

• = A ✱ = B o = C



Notice there are 2 choices as to where to put 2nd, 3rd layer of spheres
If you don't use twice in a row, you can describe any sphere packing w/ letters. e.g: ABACBABC

Hexagonal: ABAB... ABCABC: Face-centered cubic

Solved by Hales – a complicated computational proof, very difficult.

Describe ABCABC packing:

Center of spheres consist of all points $(n_1\ n_2\ n_3) \in \mathbb{R}^3$ s.t all $n_i \in \mathbb{Z}$ and $\sum n_i \cong 0 \pmod 2$

Nothing known about $n=4,\dots,7$.

Solution to $n=8$: Maryna Viazovska (Fields Medal 2022)

E_8 packing: Centers of spheres given by taking all

$(n_1\ n_2 \cdots n_8) \in \mathbb{R}^8$ All $n_i \in \mathbb{Z}$ and $\sum n_i \cong 0 \pmod 2$

$(n_1+\frac{1}{2}\ n_2+\frac{1}{2} \cdots n_8+\frac{1}{2})$ All $\sum n_i \cong 0 \pmod 2$

i.e, $\Gamma_8 = \{(n_i) \in \mathbb{Z}^8 \cup (\mathbb{Z}+\frac{1}{2})^8 : \sum n_i \cong 0 \pmod 2\}$

Q: How many spheres does each sphere touch?

Lemma: Every vector in E_8 has even squared length

Let $\vec{n} \in E_8$. $\|n\|^2 = n_1^2 + \dots + n_8^2$

Case 1: $n_i \in \mathbb{Z}$ and $\sum n_i$ EVEN

$\|n\|^2 = \sum n_i^2 \cong \sum n_i \pmod 2$

But $\sum n_i$ is even, so $\sum n_i^2$ is also even.

Case 2: x has all half-integer coordinates

$x_i = n_i + \frac{1}{2}$, $n_i \in \mathbb{Z}$, $\sum x_i = \sum(n_i+\frac{1}{2}) = \sum n_i + 4 \Rightarrow \sum n_i$ even

$\|x\|^2 = \sum (n_i+\frac{1}{2})^2 = \sum(n_i^2+n_i+\frac{1}{4}) = \sum n_i^2 + \sum n_i + 2$

Since $n_i^2 \cong n_i \pmod 2 \Rightarrow \sum n_i^2 \cong \sum n_i \pmod 2$

$\Rightarrow \sum n_i^2$ also even $\Rightarrow$ EVEN + EVEN + 2 = EVEN $\Rightarrow \|x\|^2 \cong 0 \bmod 2$.

Since $\|x\|^2$ is a positive integer even $\Rightarrow \|x^2\| \in \{2,4,6,\dots\}$

$\Rightarrow$ Shortest possible squared length is at least 2.

E.g: e_1+e_2, $e_1 \in \mathbb{R}^8$, $e_2 \in \mathbb{R}^8$

$(\dots 0 \dots \pm 1 \dots 0 \pm 1 \dots 0) \Rightarrow \binom{8}{2} \cdot 2^2 = 112$

Distance $\sqrt{2}$.

$(\pm\frac{1}{2}\ \pm\frac{1}{2} \cdots \pm\frac{1}{2}) \Rightarrow 2^8/2 = 128 \Rightarrow 240$ spheres touching each.

More generally, how many spheres at distance $\sqrt{2n}$ from O ? (C_n)

Define $\Theta(\tau) = E_4(\tau) = \sum_n C_n q^n$, $q = e^{2\pi i \tau}$ (modular form)

$$= 1 + 240 \sum_n \sigma_3(n) q^n, \text{ where } \sigma_3(n) = \sum_{d|n} d^3$$

Property 1: $\Theta(\tau+1) = \Theta(\tau)$ b/c true for $e^{2\pi i \tau} = q$ and Θ a f'n of q.

Property 2: $\Theta(-\frac{1}{\tau}) = \tau^4 \Theta(\tau)$

Poisson Summation. $L \subseteq \mathbb{R}^n$ lattice

$\sum_{\lambda \in L} f(\lambda) = \frac{1}{|L|} \sum_{\lambda \in L'} \hat{f}(\lambda)$, where L' is the dual of L, i.e consists of all vectors w/ integral inner product w/ everything in L

$|L|$ tells you density of lattice points & the volume of a F.D.

$\hat{f}(y) = \int_{\mathbb{R}^n} e^{2\pi i x y} f(x)\, dx$ is the Fourier Transform.

If $f(x) = e^{\pi i x^2 \tau} \Rightarrow \hat{f}(x) = \tau^4 e^{-\pi i x^2/\tau}$ (Shown by computation and COV)

E_8 is self dual $|L'| \cdot |L| = 1$.

Then, $\sum_{\lambda \in L} f(\lambda) = \sum e^{\pi i \tau \lambda^2} = \Theta(\tau)$ Here $L = L'$

and $\sum_{\lambda \in L} \hat{f}(\lambda) = \sum_{\lambda \in L} \tau^4 e^{-\pi i \lambda^2/\tau} = \tau^4 \sum_{\lambda \in L} e^{-\pi i \lambda^2/\tau} = \tau^4 \Theta(-1/\tau)$

$\Rightarrow \Theta(\tau) = \tau^4 \Theta(-1/\tau)$

Need P.S formula to find sphere packing bounds.

Cohn-Elkies: gives upper bound on density of sphere packing.

Suppose: ① $f(x) \leq 0 \quad |x| \geq 1 \quad x \in \mathbb{R}^n$

② $\hat{f}(x) \geq 0 \ \forall x$

Then: density of sphere packing $\leq \frac{f(0)}{\hat{f}(0)} \cdot V(\text{ball radius } 1/2)$

Special case: Centers of spheres lie on lattice L min norm ≥ 1

$f(0) \geq \sum_{x \in L} f(x)$ by supposition 1

$\Rightarrow f(0) \geq \sum_{x \in L} f(x) = \frac{1}{|L|} \sum_{x \in L'} \hat{f}(x) > \frac{1}{|L|} \hat{f}(0)$ by supposition 2

$\Rightarrow f(0) \geq \frac{1}{|L|} \hat{f}(0) \Rightarrow |L| \geq \frac{f(0)}{\hat{f}(0)}$ a nice upper bound for density of sphere packing.

Goal: To find upper bound for sphere packing, you want to find a nice function f and calculate f and its Fourier Transform

Take $f(x) = e^{-dx^2} \cdot$ (polynomial in x^2)

Elkies: Use L.P to find best polynomial and found UB for lattices.
↳ Really strange, difficult, jaggedy line to find best lattice.

Note: Best lattice for dimension 24 is Leech lattice

Showed best in $d=8$ or 24 is E_8 or Leech lattice.

Question: Can we find magic function f in dim 8 s.t the upper bound = lower bound in the E_8 / Leech lattice.

Problem: Find g on $\mathbb{R}^8$ w/ ① $g(x) \leq 0$ for $|x| \geq \sqrt{2}$ (norm min) Renorm Cohn-Elkies
② $\hat{g}(x) \geq 0$ $g(0) = \hat{g}(0) = 1$

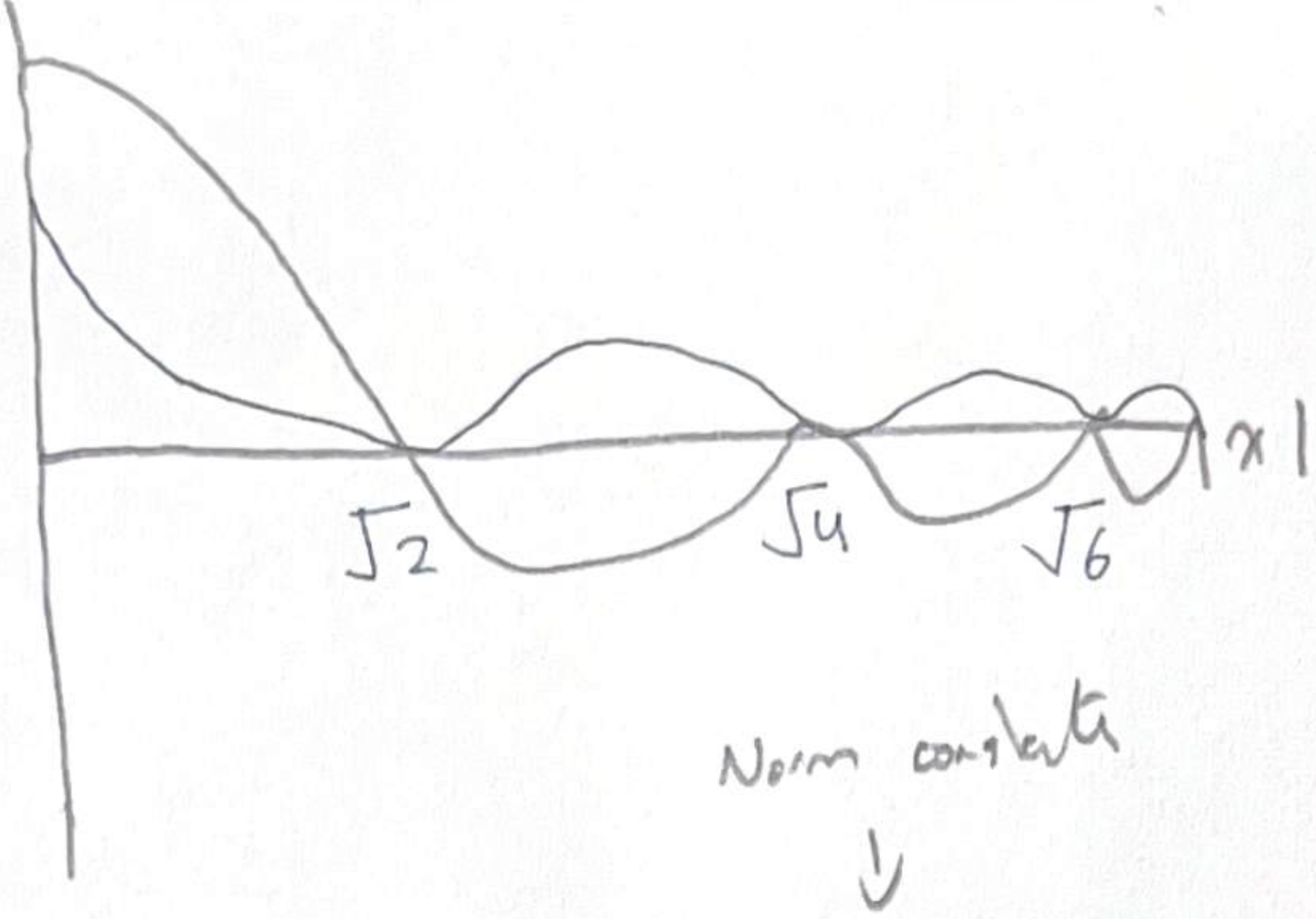
Assumptions: ① g depends only on $|x|$ i.e invariant under rotation
② $g(x) = \hat{g}(x)$ for $x \in E_8$

From pf of Cohn Elkies, "Best Case Scenario"

$f(0) \geq \sum_{x \in L} f(x)$ vanishes for lattice values $\Rightarrow f(0) = 0$

$\hat{f}(0) \leq \sum_{x \in L'} \hat{f}(x)$ vanishes in lattice not 0 $\Rightarrow$ equality

Picture: g
$\hat{g}$



To satisfy requirements

Norm constants
↓

Viazovska created: $g(x)=\frac{\pi i}{8640}a(x)+\frac{i}{240}b(x)$

$a(x)$ is even under FT i.e $\hat{a}=a$

$b(x)$ is odd under FT i.e $\hat{b}=-b$

L.T.

radius

Write $a(r)=-4\sin(\pi r^2/2)^2\int_0^{i\infty} z^2 e^{\pi i r^2 z}\phi_0\left(-\frac{1}{z}\right)dz$

($r>\sqrt{2}$ conv)

Interesting

quasi-modular form (e.g eisenstein & derivative)

$\Rightarrow \hat{a}=a,\ \hat{b}=-b$

to be computed

$\sin(x)=0 \iff x=k\pi,\ k\in\mathbb{Z}$

$\sin\left(\frac{\pi r^2}{2}\right)=0 \iff \frac{\pi r^2}{2}=k\pi \iff r^2=2k \Rightarrow r=\sqrt{2k},\ k=0,1,2,\ldots$

Correspond exactly to our squared lengths 2, 4, 6 etc... i.e r^2 even.

$b(r)=-4\sin(\pi r^2/2)^2\int_0^{i\infty} e^{\pi i r^2 z}\phi(z)\,dz$

The behavior of a and b under Fourier transforms comes from quasi-mod.

Recall quasi-modular forms are holomorphic functions whose transformation under a modular group is a polyn. in a certain term.

(neg weight mod forms w/ pole order ≤ 2 at ∞. & Elliptic mod f'n)

$$\varphi_0=\frac{1728E_4E_2^2}{E_4^3-E_6^2}+\frac{2E_2(-1728)E_4E_6}{E_4^3-E_6^2}+\frac{1728E_4^3}{E_4^3-E_6^2}-1728$$

$E_2=1-24\sum\sigma_1(n)q^n$ Standard Eisenstein series

$E_4=1+240\sum\sigma_3(n)q^n$ Combinations of Eisenstein are also

$E_6=1-504\sum\sigma_5(n)q^n$ Pretty standard

ϕ_0 has vanishing coefficients of q^{-2}, q^{-1}, q^{0} (cancel out).

Almost mod. form $\tau \to \frac{a\tau+b}{c\tau+d}$ almost invariant.

$$\psi = 128\left(\frac{\theta_{00}^4+\theta_{01}^4}{\theta_{10}^8} + \frac{\theta_{01}^4-\theta_{10}^4}{\theta_{00}^8}\right)$$

$$\theta_{00} = \sum_n q^{n^2/2}, \quad \theta_{01} = \sum (-1)^n q^{n^2/2}, \quad \theta_{10} = \sum_n q^{(n+\frac{1}{2})^2/2}$$

Rest of proof is calculation.

① Check $\hat{a} = a$, $\hat{b} = -b$

② Check positivity conditions for g and $\hat{g}$ F.T

Generalized quickly afterwards for the leech lattice

$\theta_{00}, \theta_{01}, \theta_{10}$ are the Jacobi θ fns (θ evaluated at $z=0$)

00 ⇒ no shift no sign, 01 ⇒ sign $(-1)^n$, 10 ⇒ shift $n \mapsto n+\frac{1}{2}$