Dedekind eta function

$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1-q^n)$ where $q = e^{2\pi i \tau}$, $\tau \in \mathbb{H}$

① Holomorphism

$\tau = x + yi \ (y>0)$, $q = e^{2\pi i (x+yi)} = e^{2\pi i x} \cdot e^{-2\pi y}$, $|q| = e^{-2\pi y} < 1$
$(\because y>0)$

$$\ln(\eta(\tau)) = \ln\left(q^{1/24} \prod_{n=1}^{\infty} (1-q^n)\right)$$

$$= \ln(q^{1/24}) + \ln\left(\prod_{n=1}^{\infty} (1-q^n)\right)$$

$$= \frac{1}{24}\ln(q) + \sum_{n=1}^{\infty} \ln(1-q^n) \quad * \ln(q) = \ln(e^{2\pi i \tau}) = 2\pi i \tau$$

$$= \frac{\pi i \tau}{12} + \sum_{n=1}^{\infty} \ln(1-e^{2\pi i n\tau})$$

Use $\log(1-z)$ Taylor series expansion.

$\log(1-z) = -\sum_{k=1}^{\infty} \frac{z^k}{k}$ for $|z|<1$

$$= \sum_{n=1}^{\infty}\left(-\sum_{k=1}^{\infty} \frac{(e^{2\pi i n\tau})^k}{k}\right) = -\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{e^{2\pi i k n\tau}}{k}$$

$$\ln(\eta(\tau)) = \frac{\pi i \tau}{12} - \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{q^{nk}}{k}$$ → Show convergence.

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \left|\frac{q^{nk}}{k}\right| \le \frac{1}{1-|q|} \sum_{k=1}^{\infty} \frac{|q|^k}{k} = \frac{-\ln(1-|q|)}{1-|q|} < \infty$$

$\{\tau = x+iy \mid y \ge \delta > 0\}$

Now, Assume that τ lies in a compact subset $K \subset \mathbb{H}$.
Since y is bounded by δ, it follows that $|q| = e^{-2\pi y} \le e^{-2\pi\delta}$
$\sum_{n=1}^{\infty} |q|^n$ converges.

② Transformation

$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$\eta(\tau+1) = e^{\pi i/12}\eta(\tau)$ Note that $e^{2\pi i(\tau+1)/24} = e^{\pi i\tau/12} \cdot e^{\pi i/12}$

$e^{2\pi i(\tau+1)} = e^{2\pi i\tau} = q$

$$\eta(\tau+1) = e^{2\pi i(\tau+1)/24}\prod_{n=1}^{\infty}(1-e^{2\pi i(\tau+1)n})$$

$$= e^{\pi i/12} \cdot e^{2\pi i\tau/24}\prod_{n=1}^{\infty}(1-e^{2\pi i\tau n}) = e^{\pi i/12}\eta(\tau)$$

$\eta(-\frac{1}{\tau}) = e^{-\pi i/4}\sqrt{-i\tau}\,\eta(\tau)$

Proof sketch

Express $\eta(\tau)$ as theta function

$$\eta(\tau) = q^{1/24}\sum_{n=-\infty}^{\infty}(-1)^n q^{(3n^2-n)/2}$$

Apply Poisson Summation and use modular prop. of theta function

③ Cusps $(\infty, 0)$

q expansion $\eta(\tau) = q^{1/24}(1-q+q^2-q^5+\cdots)$

There's no pole, since there are no negative powers of q.

Every rational cusp is modularly equivalent to the cusp at infinity.

$\therefore$ $\eta(\tau)$ is weight $\frac{1}{2}$ modular form with multiplier.

① Partition

ex. $P(0) = 1$

$P(1) = 1$ (1)

$P(2) = 2$ $(2, 1+1)$

$P(3) = 3$ $(3, 2+1, 1+1+1)$

$P(4) = 5$ $(4, 3+1, 2+2, 2+1+1, 1+1+1+1)$

$\vdots$

Generating Function of Partition

$$P(q) = 1 + 1 \cdot q^1 + 2 \cdot q^2 + 3 \cdot q^3 + 5 \cdot q^4 + \cdots$$

$$= \prod_{k=1}^{\infty} (1 + q^k + q^{2k} + q^{3k} + \cdots) = \underbrace{(1 + q^1 + q^2 + \cdots)}_{\#\text{ of 1's}} \underbrace{(1 + q^2 + q^4 + \cdots)}_{\#\text{ of 2's}} \underbrace{(1 + q^3 + q^6 + \cdots)}_{\#\text{ of 3's}} \cdots$$

$$= \prod_{k=1}^{\infty} \frac{1}{1 - q^k}$$

$$S = 1 + q^k + q^{2k} + \cdots \Rightarrow (1 - q^k) S = (1 - q^k)(1 + q^k + \cdots)$$

$$= 1$$

$$S = \frac{1}{1 - q^k}$$

Dedekind eta-function

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad q = e^{2\pi i \tau}$$

This is weakly modular.

Modular form $f(\tau)$ of weigh k.

$f(\gamma\tau) = (c\tau + d)^k f(\tau)$, $\forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$

but $\eta(\gamma\tau) = (c\tau + d)^{1/2} \eta(\tau)$ (extra phase)

④ $\prod_{n=1}^{\infty}(1-q^n) = q^{-1/24}\eta(\tau)$

$P(q) = \prod_{n=1}^{\infty}\frac{1}{1-q^n} = \left(\prod_{n=1}^{\infty}(1-q^n)\right)^{-1} = q^{1/24}\eta(\tau)^{-1}$

Dedekind η func Actions.

$\eta(\tau+1) = e^{\pi i/12}\eta(\tau)$

$\eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau)$ weight $1/2$

"Because the partition generating function is essentially the reciprocal of the Dedekind eta-function, it naturally inherits the modular transformation properties of $\eta(\tau)$. As a result, the partition numbers exhibit deep arithmetic structures, such as the Ramanujan congruences."

Ramanujan Congruence

$p(5n+4) \equiv 0 \pmod 5$

Why? $P(q) = p(0) + p(1)q^1 + p(2)q^2 + \cdots$

But we want $p(4)q^4 + p(9)q^9 + p(14)q^{14}$

How do we read it out once every 5 slots?

Need to think $P(q^5)$

$P(q) = q^{1/24}\eta(\tau)^{-1} \Rightarrow P(q^5) = q^{5/24}\eta(5\tau)^{-1}$ $(\because q = e^{2\pi i(5\tau)})$

To see residue classes of mod 5 $\frac{P(q)}{P(q^5)} = \frac{q^{1/24}\eta(\tau)^{-1}}{q^{5/24}\eta(5\tau)^{-1}} = q^{-1/6}\frac{\eta(5\tau)}{\eta(\tau)}$

$\left(\frac{P(q)}{P(q^5)}\right)^6 = q^{-1}\underbrace{\left(\frac{\eta(5\tau)}{\eta(\tau)}\right)^6}_{:=F(\tau)} = q^{-1}F(\tau)$

Why 6?

Ramanujan Congruence

$p(5n+4) \equiv 0 \pmod 5$ . . . $p(7n+5) \equiv 0 \pmod 7$, $p(11n+6) \equiv 0 \pmod{11}$

Proof

Key point $\quad P(q) = q^{1/24}\eta(\tau)^{-1}$

$F(\tau) = \dfrac{\eta(5\tau)^r}{\eta(\tau)^s}$ choose r, s that make $F(\tau)$ weight 0 modular form.

why?

We must compare with the constant function whose weight is 0.

To apply sturm bound, the two modular forms must have the same weight (need to compare with constant f)

The Hardy-Ramanujan asymptotic formula.

Used $\eta(-1/\tau) = e^{-\pi i/4}\sqrt{i\tau}\,\eta(\tau)$

Modular discriminat.

$\Delta(\tau) = q\prod_{n=1}^{\infty}(1-q^n)^{24} \quad q = e^{2\pi i\tau}.$

$= \eta(\tau)^{24}$

$\eta(\tau+1)^{24} = e^{2\pi i}\eta(\tau)^{24} = \eta(\tau)^{24}$, $\eta(-1/\tau)^{24} = (\sqrt{-i\tau}\,\eta(\tau))^{24}$

$= \tau^{12}\eta(\tau)^{24}$

Weight 12 modular form.

The q-expansion of $\Delta(\tau) = q - 24q^2 + 252q^3 - 1472q^4 + \cdots$

Coefficient $\tau(n)$: Ramanujan tau func.

$\tau(mn) = \tau(m)\tau(n)$ (m, n coprime)