11/24/2025

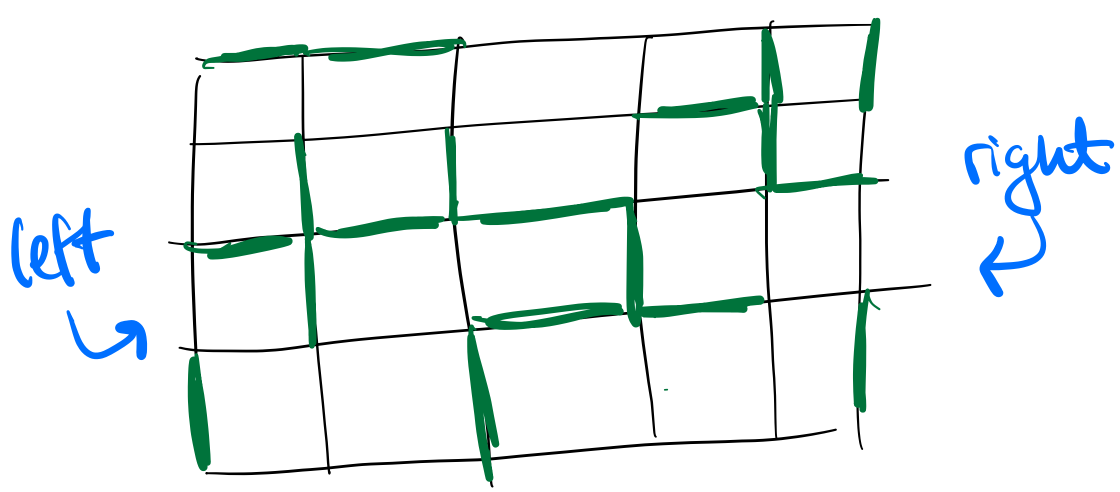
Percolation:

Consider a lattice in a complex plane.

For percolation, we have vertices (sites) & edges (bonds) between them. We assign a coloring (black or white) to sites (or bonds).

Each site is black IID w/p $p \in (0,1)$.

Consider such a finite lattice on a rectangle, with some mesh size.

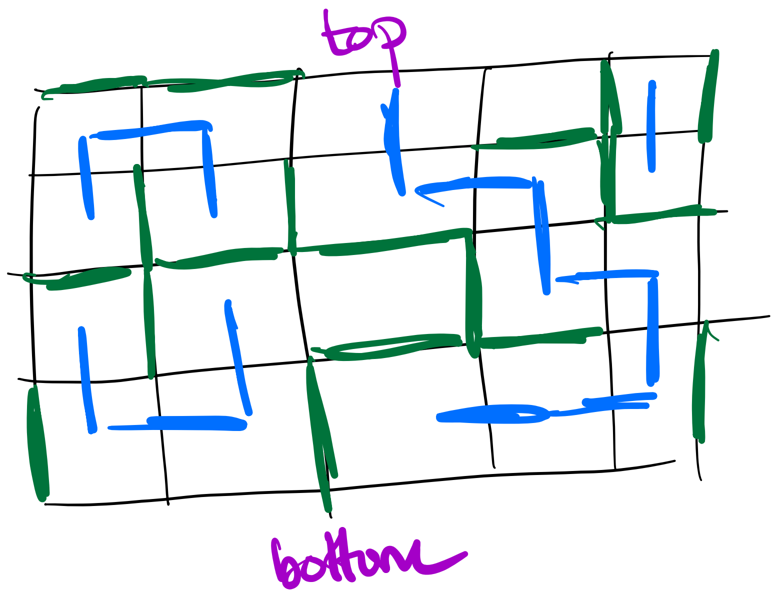


Crossing probability: as mesh $\to 0$,

$$\mathbb{P}(\text{black cluster connecting left + right}).$$

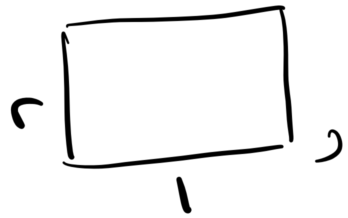
Dual lattice: place vertex on each face of lattice, each dual edge crosses one regular edge.

black $\longleftrightarrow$ white



There is always a black cluster connecting left & right sides or a white cluster connecting the bottom & top sides.

If $\Pi_p(r) :=$ crossing probability for



then $\Pi_p(r) + \Pi_{1-p}\left(\frac{1}{r}\right) = 1.$

For well-behaved lattices, $\exists!\ p_c$ s.t.

If $p < p_c$, limiting crossing prob is 0

If $p > p_c$, limiting crossing prob is 1.

Since $\Pi_{1/2}(1) = \frac{1}{2}$, $p_c = \frac{1}{2}$ for square lattice.

More generally, how to study critical probability

crossing probabilities?

At p_c, crossing prob. becomes <u>conformally invariant</u> (hol + preserves angles).

How can we think more generally abt the crossing probability? The conformal class of a quadrilateral is represented by the cross ratio of its 4 boundary pts.

Consider (z_1, z_2, z_3, z_4) under a Linear Functional Transformation

$$z \mapsto \frac{az+b}{cz+d},$$

ϕ which sends $z_2 \mapsto 1$, $z_3 \mapsto 0$, $z_4 \mapsto \infty$. Then,

$$(z_1, z_2, z_3, z_4) = \phi(z_1),$$

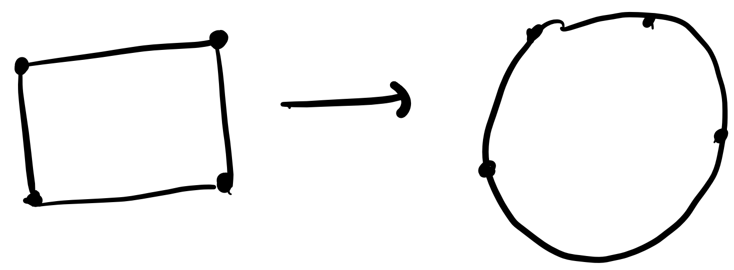
$$= \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}.$$

This mapping is unique. Notice for

$$(a,b,c,d) = (0, 1, ir+1, ir) = \frac{1}{r^2+1},$$

not conformally invariant.

<u>Change definition</u>: cross ratio should be equal to the cross ratio of any domain with 4 marked pts on boundary, if they can be mapped conformally to the vertices of rectangle.



Riemann Mapping Theorem.

Define $R_{0,1,0,r} := \{z \in \mathbb{C} : 0 \le Re(z) \le 1,\ 0 \le Im(z) \le r\}$.

Let f_r be a conformal mapping from $R_{0,1,0,r}$ to closure of $D = \{z \in \mathbb{C} : |z| < 1\}$.

We may compose this with a linear fractional transformation $\bar{D} \to \bar{H}$.

Lemma 47: There is a unique conformal map $f_r(z)$ from $R_{0,1,0,r}$ to $\bar{H}$ with $f_r(w_0) = z_0$, $f_r(w_1) = z_1$, $f_r(w_2) = z_2$, preserving cyclic order of distinct boundary pts $w_0, w_1, w_2 \in \partial R_{0,1,0,r}$, $z_0, z_1, z_2 \in \partial \bar{H}$.

Pf Sketch: Since we can map R to $\bar{D}$, suffices to show we can map $\bar{D}$ to $\bar{H}$, fixed by 3 pts

By picking 3 pts from unit disc and sending them to $\mathbb{R} \cup \infty$, we know $\partial \bar{D} \mapsto \mathbb{R} \cup \infty$ (LFT maps circle/lines to circle/lines).

Then to preserve connectedness, $\bar{D}$ goes to either upper or lower half plane. If lower, easy to compose w/ rotation that takes lower half plane to upper. $\square$

Now, let us denote a conformal map $R_{0,1,0,r}$ onto $\bar{H}$ by $\phi_r : R_{0,1,0,r} \to \bar{H}$ s.t. $\phi_r(0) = \infty$.

By def,
$$(ir, 0, ir+1) = (\phi_r(ir), \phi_r(0), \phi_r(ir+1), \phi_r(1))$$
$$= \Phi(\phi_r(ir)),$$

where Φ takes $\phi_r(0), \phi_r(ir+1), \phi_r(1)$ to $1, 0, \infty$.

We now seek to analytically extend this cross ratio:

$$\Psi : H \to \mathbb{C} \quad \text{s.t.}$$
$$\Psi(ir) = (\phi_r(ir), \phi_r(0), \phi_r(ir+1), \phi_r(1)).$$

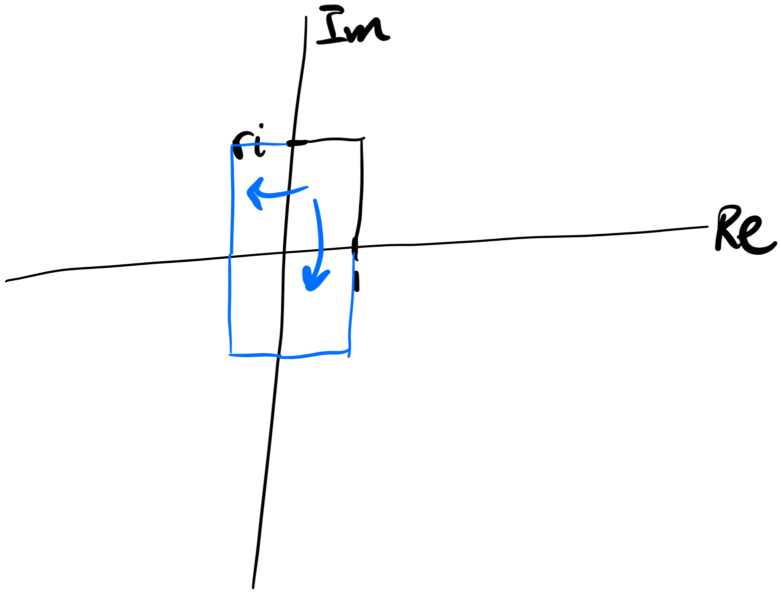
1. Apply Schwarz reflection:

$$\psi(\bar{z}) = \overline{\psi(z)}.$$

Extend domain of ϕ_r to

$$R_{0,1,-r,0} = \{z \in \mathbb{C} : 0 \leq Re(z) \leq 1, \ -r \leq Im(z) < 0\}.$$

Repeat the same argument across Re axis



We thus tile the entire $\mathbb{C}$, defined for ϕ_r.

Lemma 50: The extension of ϕ_r is an elliptic fn for $r > 0$.

Pf. By reflection, $\phi_r(z) = \phi_r(z + 2ir) = \phi_r(z+2)$.

Thus, elliptic on lattice $L(2, 2ir)$.

For this lattice, let

$$\wp_L(z) = \frac{1}{z^2} = \sum_{\omega \in L\setminus\{0\}} \left[\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right]$$

be its Weierstrass function.

Since ϕ_r, $\wp_L$ are both elliptic with same lattice & only have poles at lattice pts,

$$\phi_r(z) = a\,\wp_L(z) + b.$$

So, cross ratio for ϕ_r, $\wp_L$ same.

Corresponding pts:

$$\left.\begin{array}{lcl} 0 & \longleftrightarrow & 0 \\ 1 & \longleftrightarrow & \omega_1/2 \\ ir & \longleftrightarrow & \omega_2/2 \\ 1+ir & \longleftrightarrow & (\omega_1+\omega_2)/2 \end{array}\right\} \begin{array}{l} \wp_L(\omega_1/2) =: e_1 \\ \wp_L(\omega_2/2) =: e_2 \\ \wp_L\left(\frac{\omega_1+\omega_2}{2}\right) =: e_3, \end{array}$$

$$\wp_2(0) = \infty.$$

Fact: modular lambda function is defined by

$$\lambda(\tau) = \frac{e_3 - e_2}{e_1 - e_2}$$

for $\tau := \frac{\omega_2}{\omega_1}$. So,

cross ratio of rectangle $= \lambda(ir)$.

That is,

$$\boxed{\text{the conformally invarient cross ratio for } 1\times r \text{ rectangles is } \lambda(ir).}$$