Modular Form Undergraduate Seminar Fall 2025

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If you spot any errors, please let me know.

In the last talk, we discussed the group action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \circ z = \frac{az+b}{cz+d}.$$

Out of notational simplicity, we will often drop the $\circ$ when it is clear and notate the group action $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z$ or even just $\begin{pmatrix} a & b \\ c & d \end{pmatrix} z$.
In particular, the $\mathrm{SL}_2(\mathbb{R})$ action restricts to a group action $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$.
We also defined modular forms for $\mathrm{SL}_2(\mathbb{Z})$. In particular, we had the transformation property

$$f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \circ z\right) = (cz+d)^k f(z)$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.
To help study the group action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$, it will be helpful to understand the geometry of the quotient $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$.

2.1 Fundamental Domain

Definition 2.1. *A* ***fundamental domain*** *for the action of* $\mathrm{SL}_2(\mathbb{Z})$ *on* $\mathbb{H}$ *is a connected open subset* D *of* $\mathbb{H}$ *such that*

- *No two points of* D *are equivalent under* $\mathrm{SL}_2(\mathbb{Z})$*, and*
- $\mathbb{H}$ *can be expressed as a union of translates of the closure of* D *by elements of* $\mathrm{SL}_2(\mathbb{Z})$.

We now will describe the fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$. To do so, it is convenient to define two specific elements of $\mathrm{SL}_2(\mathbb{Z})$:

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Note that $Sz = -1/z$, and $Tz = z+1$.
What does the fundamental domain look like? It turns out to be the open set

$$D = \{z \in \mathbb{H} : |z| > 1, -1/2 < x < 1/2\}.$$

I will not try to LaTeX a picture of the fundamental domain in these notes.

Theorem 2.2 (Theorem 2.12 in Milne)**.** *The following are true:*

1. D *is a fundamental domain for* $\mathrm{SL}_2(\mathbb{Z})$.
2. *For distinct* z *and* z' *in the closure of* D*, they are* $\mathrm{SL}_2(\mathbb{Z})$*-equivalent iff*
   (a) *Re*$(z) = \pm 1/2$*, and* $z' = z \mp 1$ *(i.e.* $Tz = z'$ *or* $Tz' = z$*), or*
   (b) $|z| = 1$ *and* $z' = -1/z$.

   In particular, all such $\mathrm{SL}(2,\mathbb{Z})$*-invariant points lie on the boundary of* D.

3. *For a point $z \in \overline{D}$, the stabilizer of z is $\{\pm I_2\}$, except for*
   - $z = i$*, which has stabilizer S of order 2 in* $\mathrm{PSL}_2(\mathbb{Z})$*,*
   - $z = \rho = e^{2\pi i/6}$*, which has stabilizer TS of order 3 in* $\mathrm{PSL}_2(\mathbb{Z})$*, and*
   - $z = \rho^2$*, which has stabilizer ST^2 of order 3 in* $\mathrm{PSL}_2(\mathbb{Z})$*.*

4. $\mathrm{PSL}_2(\mathbb{Z})$ *is generated by S and T.*

Proof. Let Γ' be the subgroup of $\mathrm{SL}_2(\mathbb{Z})$ generated by S and T. We begin by showing that $\Gamma' \cdot \overline{D} = \mathbb{H}$. Consider any $z = x + iy \in \mathbb{H}$. Recall that for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma' \subseteq \mathrm{SL}_2(\mathbb{Z})$, we have that

$$\mathrm{Im}(\gamma z) = \frac{\mathrm{Im}(z)}{|cz+d|^2}.$$

Note that since $|cz+d|^2 = (cx+d)^2 + (cy)^2$, for given fixed z (i.e. fixed x and y) and positive integer N, there are finitely many (c, d) such that $|cz+d|^2 \leq N$ (In particular c must be bounded, which forces d to also be bounded). Since there are only finitely many (c, d) such that $|cz+d|^2$ is small, we conclude that there is a minimal value of $|cz+d|^2$. Thus, we conclude that there is a maximal value of $\mathrm{Im}(\gamma z)$, ranging over $\gamma \in \Gamma'$. Let γ' be this maximal choice; in other words, for any other $\gamma'' \in \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$$\mathrm{Im}(\gamma'' z) \leq \mathrm{Im}(\gamma' z).$$

Now, note that we can act via T to shift $\gamma' z$; i.e. there exist integer n such that $-\frac{1}{2} \leq \mathrm{Re}(T^n \gamma' z) \leq \frac{1}{2}$. Let $\gamma^* = T^n \gamma'$, this by definition lise in Γ'. Note that $\mathrm{Im}(\gamma^* z) = \mathrm{Im}(\gamma' z)$, as shifting by T does not change the imaginary part of an element in $\mathbb{H}$.

Now, suppose for contradiction that $|\gamma^* z| < 1$. Then acting by S on $\gamma' z$, we have that

$$\mathrm{Im}(S\gamma^* z) = \frac{\mathrm{Im}(\gamma^* z)}{|\gamma^* z|^2} > \mathrm{Im}(\gamma^* z) = \mathrm{Im}(\gamma' z),$$

contradicting the maximality of γ'. Thus, we conclude that $|\gamma^* z| \geq 1$, and so for any $z \in \mathbb{H}$, there exists $\gamma^* \in \Gamma'$ such that $\gamma^* z \in \Gamma' \cdot \mathbb{H}$.

Next, suppose that $z, z' \in \overline{D}$ are $\mathrm{SL}_2(\mathbb{Z})$-equivalent; i.e., there is such that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ such that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = z'$. WLOG, $\mathrm{Im}\, z \leq \mathrm{Im}\, z'$, and let $z = x + iy$. Then

$$\mathrm{Im}(z') = \frac{\mathrm{Im}\, z}{|cz+d|^2},$$

so we have that $|cz+d|^2 = (cx+d)^2 + (cy)^2 \leq 1$. Since $-\frac{1}{2} \leq x \leq \frac{1}{2}$ and $|y| \geq \frac{\sqrt{3}}{2}$, there are finitely many possibilities; in particular, $|c| \leq 1$. Exhausting all possibilities gives point 2, as well as finishing point 1. The group theoretic arguments that finish point 3 are left as an exercise.

For point 4, we prove the claim geometrically. We showed above that $\Gamma' \cdot \overline{D} = \mathbb{H}$. Consider any $z \in D$ (note specifically that z is not on the boundary). Consider any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$; we wish to show that $\gamma \in \Gamma'$. Since $\gamma z \in \mathbb{H}$, there exists $\gamma' \in \Gamma'$ and $z' \in \overline{D}$ such that $\gamma z = \gamma' z'$. Hence, we have that $z' = (\gamma')^{-1}\gamma z$. Since $\Gamma' \subseteq \mathrm{SL}_2(\mathbb{Z})$, $(\gamma')^{-1}\gamma \in \mathrm{SL}_2(\mathbb{Z})$, and by point 2, since $z \in D$ (and not on the boundary), we have that $z = z'$. Moreover, by point 3, since $z \in D$ is stabilized by $(\gamma')^{-1}\gamma$, we conclude that $(\gamma')^{-1}\gamma = \{\pm I_2\}$. Hence, $\gamma = \pm\gamma' \in \Gamma'$, as desired.

□

Remark 2.3. *In particular, one can show that*

$$\mathrm{PSL}_2(\mathbb{Z}) = \langle S, T : S^2 = 1, (ST)^3 = 1 \rangle.$$

Remark 2.4. *Hyperbolic geometry turns out to be a helpful way to study the above fundamental domain (and $\mathbb{H}$ in general), and could be an interesting topic for a talk. Iwaniec (Section 2.1-2.3), as listed in the references for the course, is a good reference (for a modular forms focused view of hyperbolic geometry).*

Since S and T generate $\mathrm{PSL}_2(\mathbb{Z})$, for a modular form f, the modularity condition only needs to be checked for S and T; i.e. it suffices to check that $f(-1/z) = z^k f(z)$ and $f(z+1) = f(z)$. Hence f is a periodic function. Thus, Fourier theory is useful to study f.

2.2 Fourier theory for periodic functions

I'll give a heuristic, hand-wavy description the theory for 1-periodic functions. Let $f(x) : \mathbb{R} \to \mathbb{C}$ be a 1-periodic function. The key will be functions $e^{2\pi inx}$, for $n \in \mathbb{N}$. Note that these are 1-periodic. Here, think of n as a measuring a specific frequency.
One way to get information about f is by trying to "measure" its frequency at n. This gives **Fourier coefficients**

$$a_n = \int_0^1 f(x) e^{-2\pi inx} \, \mathrm{d}x \, .$$

Here you can think of the coefficient a_n as measuring the "correlation" between f and $e^{2\pi inx}$; i.e. how similar f is to $e^{2\pi inx}$. For example, when $f(x) = e^{2\pi inx}$ exactly, then the corresponding $a_n = 1$. Hence, if f has Fourier coefficient a_n at frequency n, we see it as behaving similarly to $a_n e^{2\pi inx}$.
Moreover, for two different frequencies m and n, the exponential functions are "uncorrelated" (orthogonal). Specifically,

$$\int_0^1 e^{2\pi imx} e^{-2\pi inx} \, \mathrm{d}x = 0.$$

Since different frequencies are "uncorrelated", the information from different frequencies shouldn't "interfere" with each other. Moreover, we expect that measuring more and more frequencies gives more and more information about our function, and we can add this information together. By this logic, we can build a **Fourier series**

$$\sum_{n=-\infty}^{\infty} a_n e^{2\pi inx}$$

by adding up what we "see" the function behaving as at each frequency.
If f is sufficiently nice, the hope is that this Fourier series converges to the original function. There are far less restrictive conditions, but for our purposes, assuming f is smooth, we get the desired convergence

$$f(x) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi inx}.$$

Moreover, since f is smooth, one can show that the a_n rapidly decay in n (faster than any n^{-N}).
Details can be found in any analysis/functional analysis textbook.

2.3 Fourier expansions of modular forms

Let f be a modular form for $\mathrm{SL}(2,\mathbb{Z})$. Since f is holomorphic, it is smooth in x. Moreover, we know from before that $f(z) = f(z+1)$ (considering the transformation property under T), so f is a 1-periodic function of x. Then, we know from Fourier theory that it has a Fourier expansion of the form

$$f(x+iy) = \sum_{n=-\infty}^{\infty} b_n(y) e^{2\pi inx},$$

where $b_n(y)$ is a function of y for each n. However, since f is holomorphic, we have the Cauchy-Riemann equations

$$\frac{\partial f}{\partial y} = i \frac{\partial f}{\partial x}.$$

Applying the equations to f of the above form gives

$$\sum_{n=-\infty}^{\infty} b_n'(y)e^{2\pi inx} = \sum_{n=-\infty}^{\infty} -2\pi n b_n(y)e^{2\pi inx}.$$

Since the b_n are rapidly decaying in n, we conclude by Plancherel's theorem (theorem in functional analysis) that for these two Fourier expansions to be equal

$$b_n'(y) = -2\pi n b_n(y) \text{ for all } n,$$

and hence $b_n(y) = a_n e^{-2\pi ny}$ for some constant $a_n \in \mathbb{C}$.

Remark 2.5. *If you haven't heard of Plancherel's theorem, think of it as a statement of uniqueness of Fourier series given that the coefficients are sufficiently small.*

Thus, we can write

$$f(z) = \sum_{n=-\infty}^{\infty} a_n e^{-2\pi ny} e^{2\pi inx} = \sum_{n=-\infty}^{\infty} a_n e^{2\pi inz}.$$

It is customary to write $q = e^{2\pi iz}$. Hence we can write

$$f(z) = \sum_{n=-\infty}^{\infty} a_n q^n.$$

Remark 2.6. *If f were not holomorphic and merely real-analytic (i.e. had power series representations at all points), then these $b_n(y)$ could genuinely depend on y. For example, when f is real analytic, smooth, invariant under $\mathrm{SL}(2,\mathbb{Z})$, and has sufficient decay properties (a Maass form), the $b_n(y)$ would be Whittaker functions.*

Note that for z in the fundamental domain, we have that $0 < |q| < 1$; moreover, this function is holomorphic in this range. It is natural to want to extend to holomorphicity at $q = 0$ – note that $z \to i\infty$ gives that $q \to 0$. Hence, if f, as a function of q is holomorphic for $q = 0$, then we say that f is **holomorphic at** ∞.

Remark 2.7. *One can also see this from the Riemann surface view (to be discussed next week); this is equivalent to adding the point of infinity and defining a complex structure at the point at infinity.*

Thus, for a modular form f (i.e. for f is holomorphic at ∞), we can write the **q-expansion**

$$f(z) = \sum_{n=0}^{\infty} a_n q^n.$$

We call the a_n the **Fourier coefficients of** f. By abuse of notation, we let $f(\infty) = a_0$. In particular, if $a_0 = 0$, we say that f is a **cusp form**.

Remark 2.8. *The point at ∞ can be interpreted geometrically as a cusp (from the hyperbolic geometry point of view) – hence the terminology cusp form.*

Big idea for the rest of the semester: The Fourier coefficients will contain lots of "arithmetic information" about the modular forms.