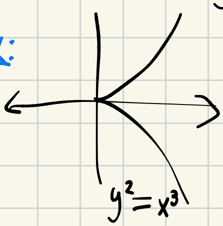
Algebraic Geometry Bootcamp

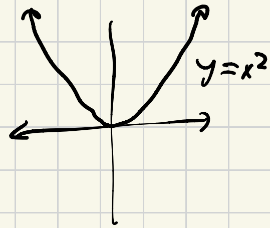
Q: What is Algebraic Geometry?

A: (Classical) Algebraic Geometry is the study of varieties, which are the zero-locus of some set of polynomials

Ex:



In highschool algebra, we think of this as dependent on 1 variable, ie given x we know what y is. Here, we take a different perspective - this is a subset of $\mathbb{R}^2$, cut out by $F(x,y)=0$, for $F(x,y) := y^2 - x^3$

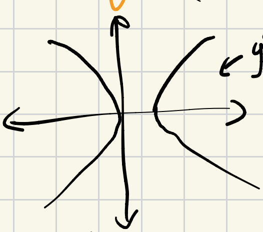


From now on, we'll work over some fixed field k_0

Def: $V \subset \mathbb{A}^n_{k_0} := k_0^n$ is an (affine algebraic) variety (over k_0) if $\exists f_1, \dots, f_m \in k_0[x_1, \dots, x_n]$ s.t. $V = \{a \in \mathbb{A}^n_k \mid f_1(a) = \dots = f_m(a) = 0\}$

Fact: There is a natural topology on $\mathbb{A}^n_{k_0}$ called the zariski topology - we can talk about "connected" varieties even if k_0 doesn't "come w/" a topology. In the zariski topology, $V \subset \mathbb{A}^n_k$ closed $\Longleftrightarrow$ V a variety

Warning!: (Zariski) connected isn't the same as real connected



$y^2 = x(x-1)$ clearly not real connected, but $y^2 - x(x-1)$ doesn't factor in $\mathbb{R}[x,y]$ (it's irreducible) which can be shown $\Rightarrow$ connected

Now, lets examine X some connected variety over k_0. In geometry (or really math in general), it is often easier to study the fns on some geometric object, rather than the object itself - because of this, we'll examine a certain class of fns on our variety, called regular functions.

Def: $f: V \xrightarrow{\subset \mathbb{A}^n_{k_0}} k_0$ is regular if its the restriction of a polynomial fn $f': \mathbb{A}^n_{k_0} \to k_0$

We call $\mathcal{O}(V) := \{\text{regular fns on } V\}$ V's coordinate ring. Because this is a ring, we can use algebraic tools on it!

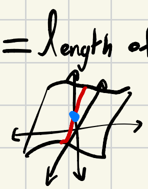
Fact: Rings have a notion of (Krull) dimension, which corresponds to the maximal length of chains of prime ideals (ie $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \dots \subsetneq \mathfrak{p}_n \subset R \leftarrow \dim(R) = \max_{\text{chains}} n$)

Def For a variety X, we say $\dim(X) = \dim \mathcal{O}(X) = \text{tr deg. Frac}\, \mathcal{O}(X)/k_0$

nontrivial - proved in Comm Alg last class (requires a half sem of comm alg)

Why do we use this definition?

If k_0 alg closed, this dimension = length of chain of irred subvarities (ie a surface contains a curve contains a pt)

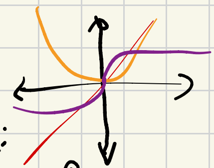


but this lets us talk about dimension in a much more general sense

Warning 2: If you study AG, you'll quickly see that affine vtys are a specific example of an algebraic variety and that many of my defs are specific to affine vtys — I'm 'lying' so that it looks familiar but know most of this isn't general / technically true

Def: An algebraic curve is an algebraic variety of dim 1

Fact: The zero locus of any single polynomial in $\mathbb{A}^2_{k_0}$ is an alg curve. Ex:



all 3

Thm (Thm 7.5 in Milne): Every cpt Riemann Surface X has a unique structure of a complete, nonsingular algebraic curve

Think of this as some 'cptness' condition - pretty much NO affine curves will be cplete

for a plane curve cut out by f, nonsingular $\iff \forall c \in X$, $\nabla f(c) \neq \vec{0}$

Pf idea) Because we've lost so much generality, this proof is hard to describe, but all we really need to do is describe a topology, and regular fns on open sets, Open sets = Sets w/ finite complement, and reg fns on U = holom fns on U meom globally

Back to $X_0(N)$!

Because $X_0(N)$ a cpt RS, by thm $\exists!$ a $\mathbb{C}$ alg-curve structure for $X_0(N)$, which we'll call $X_0(N)_{\mathbb{C}}$. By construct, we know what rtl fns on $X_0(N)_{\mathbb{C}}$ are — $\mathbb{C}(j(z), j(Nz))$ — this actually characterizes $X_0(N)_{\mathbb{C}}$ among cplete non-sing curves over $\mathbb{C}$, since the theorem tells us such a curve is unique.

By Yifan's last lecture, we know a curve w/ this field of rtl functions — the curve C defined by $F_N(X,Y)=0$, where $F_N(X,Y)$ the minimal poly (w/ integer coefficients). This curve might be singular/non-cplete, but we can 'patch' that to get $\overline{C}$, a cplete nonsingular curve whose coord fns x, y satisfy $F_N(x,y)=0$. This is a $\mathbb{C}$ cplete alg curve over $\mathbb{Q}$ (ie defined by a poly w/ $\mathbb{Q}$ coefficients) - by uniqueness this is isom to $X_0(N)_{\mathbb{C}}$, all that's left is to describe this isom $\overline{C} \cong X_0(N)_{\mathbb{C}}$, then we'll call $\overline{C} = X_0(N)_{\mathbb{Q}}$ our canonical model of $X_0(N)$ over $\mathbb{Q}$. We have a unique isom making (x,y) on $\overline{C}$ correspond to $(j(z), j(Nz))$ on $X_0(N)_{\mathbb{C}}$ (since both satisfy this minimal poly F_N) so this is what establishes our isom. $\square$