Step ⑥ : Symmetry for $N > 1$.

$F(j(z), j(Nz)) = 0$. Substitute $z$ by $\frac{-1}{Nz}$ $\Rightarrow j(z)$ by

$j(z) = j(-\frac{1}{z})$ & $j(-\frac{1}{Nz}) = j(Nz)$ since $\begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix} \in \Gamma(1)$ & $j$ invariant under $\Gamma(1)$.

Hence, we have

$$F(j(Nz), j(z)) = 0.$$

Therefore, $F(Y, X) = cF(X, Y) \Rightarrow F(X, Y) = c^2 F(X, Y) \Rightarrow c = \pm 1$.

If $c = -1$, we would have $F(X, X) = -F(X, X) \Rightarrow F(X, X) = 0$.

$\Rightarrow (X - Y) \mid F(X, Y)$ which is impossible ($j(z) \not\equiv j(Nz)$, $N > 1$).

$\Rightarrow c = 1$

Step ⑦ For $N = p$.

the conjugates $j(p\gamma_i z) = j(\frac{z+m}{p})$ for $m = 0, 1, \cdots, p-1$ when $\gamma \neq 1$

Let $\xi_p = e^{\frac{2\pi i}{p}}$. $(1 - \xi_p) \subseteq \mathbb{Z}[\xi_p]$ prime ideal. $(1-\xi_p)^p = (p)$

Consider the $q$-expansion:

$$j\left(\frac{z+m}{p}\right) = \frac{1}{e^{2\pi i (z+m)/p}} + 744 + \cdots$$

is a Laurent series in $q^{\frac{1}{p}}$ w/ coeff in $\mathbb{Z}[\xi_p]$

$$= \frac{\xi_p^{-m}}{q^{1/p}} + \cdots$$

Modulo $(1 - \xi_p)$, all $\xi_p^{-m} \equiv 1$

Hence the $q$-expansion of $j(\frac{z+m}{p})$ are identical $\bmod (1 - \xi_p)$

So. $F(j(z), Y) = (Y - j(pz)) \prod_{m=0}^{p-1} (Y - j(\frac{z+m}{p}))$

$$\equiv (Y - j(pz))(Y - j(\tfrac{z}{p}))^p \mod (1 - \xi_p).$$

But $j(\frac{z}{p}) \equiv j(z)^{\frac{1}{p}} \bmod (1 - \xi_p)$

$j(pz) \equiv j(z)^p$ — —

$$\Rightarrow F(j(z), Y) \equiv (Y - j(z)^p)(Y - j(z)^{\frac{1}{p}})^p \mod (1 - \xi_p) \text{ hence}$$

$$\equiv (Y - j(z)^p)(Y^p - j(z)) \mod p.$$

$$\equiv Y^{p+1} + X^{p+1} - X^p Y^p - XY \mod p.$$

Canonical Model of $X_0(N)$ over $\mathbb{Q}$.

Review of AG:

The main thm we proved previously will allow us to define a model of $X_0(N)$ over $\mathbb{Q}$. But to achieve that goal, we need some preparation (notions in AG.

Def: 1) A presheaf $\mathcal{F}$ on $X$ top. space is a "map" assigning to each open $U \subseteq X$ a set $\mathcal{F}(U)$ and to each $U' \subseteq U$, a "restriction map" $\mathcal{F}(U) \to \mathcal{F}(U')$ s.t.

$$a \mapsto a|_{U'}.$$

- For the inclusion $U \xrightarrow{id} U$ (i.e. $U \subseteq U$). $\mathcal{F}(U) \to \mathcal{F}(U)$ is the identity

- For $U'' \subseteq U' \subseteq U$. $\mathcal{F}(U) \to \mathcal{F}(U'')$ is the composition

$$\mathcal{F}(U) \to \mathcal{F}(U') \to \mathcal{F}(U'')$$

2). If the sets $\mathcal{F}(U)$ are abelian gps and the restriction maps are gp homom. then $\mathcal{F}$ is a presheaf of ab. gp (similarly for .. of rings, modules. etc.)

3) A presheaf $\mathcal{F}$ is a sheaf if (it satisfies the gluing conditions). For every open covering $\{U_i\}$ of $U \subseteq X$

① (Sections are determined locally). $s, t \in \mathcal{F}(U)$. if $s|_{U_i} = t|_{U_i}$ $\forall i \Leftrightarrow s = t$

② (Existence). $s_i \in \mathcal{F}(U_i)$ if $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for $\forall i, j$. then $\exists$ $s \in \mathcal{F}(U)$ s.t. $s_i = s|_{U_i}$

4). A ringed space is a pair $(X, \mathcal{O}_X)$ consisting of a top. space $X$ and a sheaf of ring $\mathcal{O}_X$ on $X$

5) Let $k_0$ be a field, $k = \overline{k_0}$, An affine $k_0$-algebra $A$ is a f.g. $k_0$-algebra $A$ s.t. $k \otimes_{k_0} A$ is reduced.

Zariski ~~topology~~ topology on Specm A

6) Let $A$ be an affine $k_0$-algebra. Let

Specm $A := \{$max ideals in $A\}$ endow it w/ top:

The set: $D(f) := \{m \mid f \notin m\}$ $f \in A$, form a basis for the open sets

There is ~~the~~ a unique sheaf of $k_0$-algebras $\mathcal{O}$ on Specm $A$ s.t. $\mathcal{O}(D(f)) = A_f$.

For $f \in A$, $v \in \mathrm{Specm}\, A$ (i.e. $m_v \subseteq A$), we set $f(v) :=$ the image of $f$ in $K(v) := A/m_v$. Then the set $V(f)$ of zeros of $f$ in $X$ is well defined $= X \setminus D(f)$.

We now have a ringed space $(\mathrm{Specm}\, A, \mathcal{O})$ called affine variety over $K_0$.

A prevariety $(X, \mathcal{O}_X)$ over $K_0$: if $X$ admits a finite covering $(U_i)$ s.t. $(U_i, \mathcal{O}_X|_{U_i})$ is affine variety.

A morphism of prevarieties over $K_0$ is a morphism of ringed spaces: i.e. when $K_0 = K$, can regard $\mathcal{O}_X$ as a sheaf of $K$-valued functions on $X$, and a morphism $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a cts. map $\varphi : X \to Y$ s.t. $f \in \mathcal{O}_Y(U) \Rightarrow f \circ \varphi \in \mathcal{O}_X(\varphi^{-1}(U))$
$\varphi^b : \mathcal{O}_Y \to f_* \mathcal{O}_X$.

Note when $V = \mathrm{Specm}\, A$, $W = \mathrm{Specm}\, B$, $\exists$ 1-1 correspondence between $W \to V$ & homom of $K_0$-algebras $A \to B$.

Base change & model

There is a canonical way of attaching a variety $X$ over $K$ to a variety $X_0$ over $K_0$: Given $X_0$ over $K_0$, $K_0 \subseteq K$, the variety $X$ is obtained by base change.

e.g. if $X_0 = \mathrm{Specm}\,(A)$, then $X = \mathrm{Specm}\,(A \otimes_{K_0} K)$

This new object $X$ is a variety over $K$, w/ $X \to X_0$ induced by $A \to K \otimes_{K_0} A$, $a \mapsto 1 \otimes a$. And $X_0$ is called a model for $X$ over $K_0$.

A point of $X$ w/ coordinates in $K_0$ or a $K_0$-point of $X$ is a morphism $\mathrm{Specm}\, K_0 \to X$.

ex If $X = \mathrm{Specm}\, A$, then a point of $X$ w/ coordinates in $K_0$ is a $K_0$-homom $A \to K_0$. By Alg II.

If $A = K[X_1, \cdots, X_n]/(f_1, \cdots, f_m)$, then to give a $K_0$-homom $A \to K_0 \Leftrightarrow (a_1, \cdots, a_n)$ s.t. $f_i(a_1, \cdots, a_n) = 0 \ \forall i \in \{1, \cdots, m\}$.

e.g. $X(\mathbb{C}) = X$ for $X$ Riemann Surface.

Write $X(K_0)$ $K_0$-points of $X$.