Recall the function $r(n,k)$:

$$r(n,k) := \#\{(m_1,\dots,m_k) \in \mathbb{Z}^k : n = m_1^2 + \cdots + m_k^2\}.$$

Then we note that

$$r(n,k) = \sum_{n_1+n_2=n} r(n_1,k_1)\, r(n_2,k_2)$$

for fixed $k_1 + k_2 = k$.

This mimics convolution for power series / polynomials.

Consider

$$\theta(z,k) = \sum_{n=0}^{\infty} r(n,k)\, e^{2\pi i z n} = \sum_{n=0}^{\infty} r(n,k)\, q^n.$$

Proposition 15:

For each $k \geq 0$, the series $\theta(z,k)$ converges absolutely and defines a holomorphic function on $\mathbb{H}$.

Proof.

Key idea, we split $r(n,k)q^n$ into $\sum q^n$ (sum $r(n,k)$ times), via

$$\theta(z,k) = \sum_{v\in\mathbb{Z}^k} e^{2\pi i |v|^2 z}.$$

Take any compact region $K\subset\mathbb{H}$ for which $\mathrm{Im}(z)\geq y_0$ for some $y_0>0$. Then, consider the tail

$$\sum_{\substack{v\in\mathbb{Z}^k,\\ |v|\geq M}} \left|e^{2\pi i|v|^2 z}\right| = \sum_{m=M}^{\infty}\sum_{\substack{v\in\mathbb{Z}^k,\\ |v|=m}} \left|e^{2\pi i m^2 z}\right|$$

split z into Re + Im

$$\leq \sum_{m=M}^{\infty}\sum_{\substack{v\in\mathbb{Z}^k,\\ |v|=m}} e^{-2\pi m^2 y_0}.$$

$$\leq \sum_{m=M}^{\infty}(2m+1)^k e^{-2\pi m^2 y_0}.$$

Pick M big enough s.t. $(2m+1)^k\leq e^m$ & $\pi m^2 y_0\geq m$ $\forall m\geq M$. So,

$$\leq \sum_{m=M}^{\infty} e^m e^{-2m} \xrightarrow{M\to\infty} 0$$

$\forall z\in K$. So, absolute & uniform converge on compact sets.

$\square$

Absolute converge means we can reorder terms:

$$\theta(z,k_1)\,\theta(z,k_2) = \left(\sum_{n_1=0}^{\infty} r(n_1,k_1)q^{n_1}\right)\left(\sum_{n_2=0}^{\infty} r(n_2,k_2)q^{n_2}\right)$$

$$= \sum_{n=0}^{\infty}\left(\sum_{n_1+n_2=n} r(n_1,k_1)\,r(n_2,k_2)\right)q^{n}$$

$$= \theta(z, k_1+k_2).$$

Note that $\theta(z,k)$ is $\mathbb{Z}$-periodic in z (since q is).

<u>Proposition 16</u>: The function $\theta(z,4)$ is modular of weight 2 for the congruence subgroup $\Gamma_0(4)$.

$$\Gamma_0(4) = \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \pmod 4\right\}.$$

Modular of weight 2 for $\Gamma_0(4)$ means

$$(cz+d)^{-2} f(\gamma(z)) = f(z)$$

$\forall z \in \mathbb{H}$, $\forall \gamma \in \Gamma_0(4)$.

Proof of Prop 16:

Weak modularity: $\Gamma_0(4)$ generated by $\pm\begin{pmatrix}1&1\\0&1\end{pmatrix}$ and $\pm\begin{pmatrix}1&0\\4&1\end{pmatrix}$. First is just $\mathbb{Z}$-translation, which θ is invariant under, so we just consider 2nd.

First we study $\theta(\frac{z}{2}, 1) = \sum_{d\in\mathbb{Z}} e^{\pi i d^2 z}$.

Consider z on imaginary axis, so $\theta(it, 1) = \sum_d e^{-\pi d^2 t}$.

$$f(d) = e^{-\pi d^2} = \sum_{m\in\mathbb{Z}} \underbrace{e^{-\pi m^2}}_{\hat{f}(m),\ \boxed{\text{fourier}}} e^{-2\pi i m d}$$

Scaling: For $f(d\sqrt{t})$, we have $\frac{1}{\sqrt{t}}\hat{f}(m/\sqrt{t})$.

Poisson Summation Formula: take $h(x) = e^{-\pi x^2 t}$. Then,

$$\theta(it, 1) = \sum_d h(d) = \sum_m \hat{h}(m) = \sum_m \frac{1}{\sqrt{t}} f\left(\frac{m}{\sqrt{t}}\right)$$

$$= \frac{1}{\sqrt{t}}\,\theta\left(\frac{i}{t}, 1\right).$$

Thus, the identity $\boxed{\theta\left(\frac{-1}{z}, 1\right) = \sqrt{-iz}\,\theta(z, 1)}$ true for all $z\in\mathbb{H}$ by Identity Thm (both hol on Im axis).

Now,

$$\begin{pmatrix}0 & 1/4\\ -1 & 0\end{pmatrix}\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & -1\\ 4 & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\ 4 & 1\end{pmatrix} = \gamma$$

$z \mapsto \frac{1/4}{-z} = \frac{-1}{4z}$ $\qquad$ $z \mapsto z-1$ $\qquad$ $z \mapsto \frac{-1}{4z}$

$$\theta\left(\frac{z}{4z+1}\right) = \theta\left(\_\_ \circ \_\_ \circ \_\_(z)\right)$$

$$= \theta\left(\left(z \mapsto \frac{-1}{4z}\right) \circ \left(\frac{-1}{4z} - 1\right)\right)$$

$$= \alpha\sqrt{i\left(\frac{1}{4z}+1\right)}\ \theta\left(\frac{-1}{z} - 4\right)$$

$$= \alpha\sqrt{i\left(\frac{1}{4z}+1\right)}\ \theta\left(\frac{-1}{z}\right)$$

$$= \alpha\sqrt{i\left(\frac{1}{4z}+1\right)}\ \sqrt{-iz}\ \theta(z)$$

$$= \sqrt{4z+1}\ \theta(z).$$

$$\theta\left(\frac{-1}{z}, 1\right) = \sqrt{-iz}\ \theta(z, 1) \longrightarrow \theta\left(\frac{-1}{4z}, 1\right) = 2\sqrt{-iz}\ \theta(4z, 1)$$

So, for $\theta(z,4) = \theta(z,1)^4$,

$$\theta(\gamma(z), 4) = (4z+1)^2 \theta(z,4)$$

$\rightarrow$ weight 2.

Is it holomorphic at cusps? Yes, by some technical details.

So, $\theta(z,4) = M_2(\Gamma_0(4))$.

Proof of Final Result:

From earlier result, $M_2(\Gamma_0(4))$ is 2-dim.

Recall $G_{2,2}$ & $G_{2,4}$:

$$G_{2,N}(z) = G_2(z) - N\, G_2(Nz).$$

both in $M_2(\Gamma_0(4))$, lin. indep by Fourier series coeffs (Prop 6). Then,

$$\theta(z,4) = a\,G_{2,2} + b\,G_{2,4}$$
$$= -a\frac{\pi^2}{3}(1+24q+\cdots) - b\pi^2(1+8q+\cdots).$$

From first two terms, $a=0$, $b=\frac{-1}{\pi^2}$. Collecting terms,

$$\theta(z,4) = \sum_{n=0}^{\infty} r(n,4)q^n = \sum_{n=0}^{\infty}\left[8\sum_{\substack{0<d\mid n,\\ 4\nmid d}} d\right]q^n.$$

<u>Reference</u>:

$$G_{2,2}(z) = -\frac{\pi^2}{3}\left(1+24\sum_{n=1}^{\infty}\left(\sum_{\substack{d\mid n\\ 2\nmid d}} d\right)q^n\right)$$
$$G_{2,4}(z) = -\pi^2\left(1+8\sum_{n=1}^{\infty}\left(\sum_{\substack{d\mid n\\ 4\nmid d}} d\right)q^n\right)$$