

Modular Forms for Other Subgroups

Sebastian Pallais-Aks

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Outline

Introduction

Fourier Expansions and Eisenstein Series

Hecke Operators and Newform Theory

The Fricke Involution

Introduction

$\Gamma(1)$ is really neat and beautiful, but this limits its applicability to much of modern number theory.

Congruence Subgroups (Review)

Modern number theory (elliptic curves, Galois representations) relies on **congruence subgroups** $\Gamma \subset \Gamma(1)$.

$$\blacktriangleright \Gamma_0(N) = \left\{ \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{N} \right\}$$

$$\blacktriangleright \Gamma_1(N) = \left\{ \gamma \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}$$

$$\blacktriangleright \Gamma(N) = \left\{ \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}$$

Introduction

Congruence Subgroups (Review)

$$\Gamma_1(N) \subset \Gamma_0(N) \subset \Gamma(1)$$

Goal: Highlight the key changes when moving from $\Gamma(1)$ to Γ .

Review: Geometry of $X(\Gamma)$ vs $X(1)$

Moving from $\Gamma(1)$ to a congruence subgroup Γ

- ▶ **Fundamental Domain:** Becomes larger (index = area ratio), potentially disconnected.
- ▶ **Modular Curve $X(\Gamma)$:** Still a compact Riemann surface, but more complex.

Review: Geometry of $X(\Gamma)$ vs $X(1)$

Key Geometric Changes

- ▶ **Genus $g \geq 0$:** Often positive (calculated via Riemann-Hurwitz), unlike $g = 0$ for $X(1)$. Ex: $X_0(11)$ has $g = 1$.
- ▶ **Multiple Cusps ($\epsilon_\infty \geq 1$):** The single cusp of $X(1)$ splits into distinct Γ -orbits.
- ▶ **Fewer Elliptic Points (ϵ_2, ϵ_3):** Only occur if Γ intersects the stabilizer in $\Gamma(1)$ non-trivially. $\Gamma(N)$ has none for $N > 1$.

Topology

Requires careful definition of neighborhoods (horocycles) at cusps to ensure that $X(\Gamma)$ is Hausdorff

Review: Algebra of Forms and Riemann-Roch

- ▶ Much more complex structure for Γ than $\Gamma(1)$
- ▶ **Riemann-Roch Theorem:** Becomes the essential tool.
 - ▶ Relates modular forms $f \in \mathcal{M}_k(\Gamma)$ to $k/2$ -fold differentials ω on $X(\Gamma)$.
 - ▶ Relates orders of vanishing $\nu(f)$ and $\nu(\omega)$ (depends on point type: elliptic, cusp, other).
 - ▶ Yields dimension formulas for $\mathcal{M}_k(\Gamma)$ and $\mathcal{S}_k(\Gamma)$ in terms of $g, \epsilon_2, \epsilon_3, \epsilon_\infty$.
 - ▶ Crucial result: $\dim(\mathcal{S}_2(\Gamma)) = g$.

$\Gamma(1)$: One Cusp, One Series

- ▶ There is only one cusp at $i\infty$. The stabilizer is generated by $z \mapsto z + 1$, or $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.
- ▶ This single periodicity means every modular form has a Fourier expansion in powers of $q = e^{2\pi iz}$.
- ▶ Fourier expansion at the cusp: $f(z) = \sum_{n=0}^{\infty} a_n q^n$
- ▶ For each even weight $k \geq 4$, the Eisenstein space $\mathcal{M}_k/\mathcal{S}_k$ is one-dimensional, spanned by the class of the classical Eisenstein series $E_k(z) = \frac{1}{2\zeta(k)} \sum_{(c,d) \neq (0,0)} (cz + d)^{-k}$.

Γ : Multiple Cusps

- ▶ A modular form for Γ has a distinct Fourier expansion at each of the ϵ_∞ cusps.
- ▶ A modular form $f \in \mathcal{M}_k(\Gamma)$ must be holomorphic at every cusp
- ▶ At a cusp $s = \alpha(\infty)$, we analyze the expansion of $f[\alpha]_k$. The periodicity at this cusp is determined by the smallest $h_s > 0$ such that $\alpha \begin{pmatrix} 1 & h_s \\ 0 & 1 \end{pmatrix} \alpha^{-1} \in \Gamma$. This h_s is the ****width**** of the cusp.
- ▶ The expansion is a series in powers of $q_{h_s} = e^{2\pi iz/h_s}$.
- ▶ We need a richer family of Eisenstein series to account for the constant terms at all these different cusps. The dimension of the Eisenstein space is ϵ_∞ (for $k \geq 3$).

Generalized Eisenstein Series

For $\Gamma(N)$:

For each cusp, represented by a vector $v \in (\mathbb{Z}/N\mathbb{Z})^2$, we define an Eisenstein series $E_k^v(\tau)$ by summing over lattice vectors in that congruence class.

$$E_k^v(\tau) = \sum_{(c,d) \equiv v \pmod{N}, \gcd(c,d)=1} (c\tau + d)^{-k}$$

This series is specifically constructed to be non-zero at the cusp corresponding to v and zero at all others.

Generalized Eisenstein Series

For $\Gamma_1(N)$ and $\Gamma_0(N)$

The space of all forms decomposes into eigenspaces for the "diamond operators". The basis for the Eisenstein series in these spaces is constructed by taking linear combinations of the E_k^χ series, weighted by Dirichlet characters.

This provides a complete and explicit basis for the Eisenstein space $E_k(\Gamma) = \mathcal{M}_k(\Gamma)/\mathcal{S}_k(\Gamma)$.

For $\Gamma(1)$: A Commuting, Hermitian Family

- ▶ The Hecke operators T_n form a commuting family of linear operators on $\mathcal{M}_k(\Gamma(1))$ and $\mathcal{S}_k(\Gamma(1))$.
- ▶ Crucially, they are Hermitian with respect to the Petersson inner product on cusp forms: $\langle T_n f, g \rangle = \langle f, T_n g \rangle$.
- ▶ By the spectral theorem, this guarantees an orthogonal basis of $\mathcal{S}_k(\Gamma(1))$ consisting of simultaneous eigenforms for all T_n .

New Operators, New Problems

Diamond Operators $\langle d \rangle$ for $d \in (\mathbb{Z}/N\mathbb{Z})^*$

A new family of operators arises from the quotient

$$\Gamma_0(N)/\Gamma_1(N) \cong (\mathbb{Z}/N\mathbb{Z})^*$$

$$\langle d \rangle f = f[\alpha]_k \text{ for any } \alpha = \begin{bmatrix} a & b \\ c & \delta \end{bmatrix} \in \Gamma_0(n) \text{ with } \delta \equiv d \pmod{N}$$

The diamond operator $\langle d \rangle$ respects the decomposition

$\mathcal{M}_k(\Gamma_1(N)) = \bigoplus_{\mathcal{X}} \mathcal{M}_k(N, \mathcal{X})$, operating on the eigenspace associated to each character $\mathcal{X}$ as multiplication by $\mathcal{X}(d)$

New Operators, New Problems

Modified Adjoint Property

The second type of Hecke operator, denoted T_p is also a weight- k double coset operator $[\Gamma_1 \alpha \Gamma_2]_k$ where $\Gamma_1 = \Gamma_2 = \Gamma_1(N)$ but now

$$\alpha = \begin{bmatrix} 1 & 0 \\ 0 & p \end{bmatrix}, \quad p \text{ prime}$$

For $p \nmid N$, the operators are no longer Hermitian but normal. They satisfy $T_p^* = \langle p \rangle^{-1} T_p$. This still guarantees an orthogonal basis of eigenforms for operators with $p \nmid N$.

The Problem of "Oldforms"

- ▶ The space $\mathcal{S}_k(\Gamma_1(N))$ contains forms "lifted" from lower levels $M|N$. For example, if $f(z) \in \mathcal{S}_k(\Gamma_1(M))$, then $f(z)$ and $f(dz)$ are in $\mathcal{S}_k(\Gamma_1(N))$.
- ▶ This creates redundancy and destroys the "multiplicity-one" property: different forms can have the same eigenvalues for T_p with $p \nmid N$. This is a major obstacle to defining canonical forms.

Atkin-Lehner Newform Theory

Oldspace $\mathcal{S}_k^{old}$

The subspace spanned by all forms lifted from levels M that are proper divisors of N .

A cusp form f lies in the oldspace if and only if its Fourier coefficients $a_n(f)$ are zero for all n coprime to the level N .

Newspace $\mathcal{S}_k^{new}$:

The orthogonal complement of the oldspace.

$$\mathcal{S}_k(\Gamma_1(N)) = \mathcal{S}_k(\Gamma_1(N))^{old} \oplus \mathcal{S}_k(\Gamma_1(N))^{new}$$

Both subspaces are stable under the full Hecke algebra $\mathbb{T}$ generated by all T_n and $\langle n \rangle$.

Atkin-Lehner Newform Theory

Multiplicity-One Theorem

The newspace $\mathcal{S}_k^{new}$ has an orthogonal basis of newforms. A newform is a normalized ($a_1 = 1$) common eigenform for the *full* Hecke algebra. For any given system of eigenvalues, there is a **unique** newform having them.

Canonical Forms and L-functions

- ▶ Newforms are the fundamental, arithmetically significant building blocks of cusp forms.
- ▶ Being an eigenform for all T_n is equivalent to its L-function having an Euler product over **all** primes:

$$L(s, f) = \sum_{n=1}^{\infty} a_n(f) n^{-s} = \prod_p (1 - a_p(f) p^{-s} + \chi(p) p^{k-1-2s})^{-1}$$

- ▶ This complete Euler product is the crucial property that connects modular forms to arithmetic geometry.
- ▶ The Modularity Theorem states that every elliptic curve over $\mathbb{Q}$ corresponds to a unique rational newform of weight 2.

A New Symmetry for $\Gamma_0(N)$

- ▶ For $\Gamma(1)$, the matrix $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is a group generator. Its action is part of the modularity condition.
- ▶ For $\Gamma_0(N)$, the matrix $W_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$ is *not* in the group, but it **normalizes** it: $W_N \Gamma_0(N) W_N^{-1} = \Gamma_0(N)$.
- ▶ This gives rise to a new operator, the **Fricke Involution**, an extra symmetry on the space of forms:

$$(W_N f)(\tau) = (N^{1/2} \tau)^{-k} f(-1/(N\tau))$$

The Fricke Involution

- ▶ The Fricke involution W_N is self-adjoint on $\mathcal{S}_k(\Gamma_0(N))$ and squares to the identity (up to sign).
- ▶ It is the principal member of a family of Atkin-Lehner involutions W_Q for each prime power $Q||N$.

The Fricke Involution

- ▶ **Key Result of Atkin-Lehner Theory:** Newforms are simultaneous eigenforms for all Atkin-Lehner involutions W_Q . The eigenvalue is always ± 1 .

$$W_Q f = \epsilon_Q f, \quad \epsilon_Q = \pm 1$$

- ▶ The sequence of eigenvalues (ϵ_Q) provides a canonical "signature" for a newform.
- ▶ The eigenvalue of the main involution W_N determines the sign in the functional equation of the newform's L-function. This sign is a critical arithmetic invariant.