We have studied the Riemann surface & modular forms associated with $SL_2(\mathbb{Z})$.

But mathematicians thought, "since $SL_2(\mathbb{Z})$ contains so many transformations, what if we consider only those that satisfy certain conditions? Wouldn't that allow us to see a more refined structure?"

Today, we'll look at certain subgroups of $SL_2(\mathbb{Z})$ that consist only of transformations satisfying specific conditions.
We'll see how the corresponding Riemann Surfaces change, also Fundametal Domain & how the modular forms defined on them behave differently.
The goal is to understand everything in an intuitive way, rather than going into detailed proofs.

Congruence Subgroups of $SL_2(\mathbb{Z})$ $(\Gamma_0(N), \Gamma_1(N), \Gamma(N))$

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod N \right\}$$

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod N \right\}$$

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{I\text{(Identity Matrix)}} \pmod N \right\}$$

As we can see intuitively,

$$SL_2(\mathbb{Z}) \supseteq \Gamma_0(N) \supseteq \Gamma_1(N) \supseteq \Gamma(N)$$

Then how does the Fundamental Domain change?

Recall: $\tau_1 \sim \tau_2$, iff $\exists \gamma \in SL_2(\mathbb{Z})$ s.t $\tau_2 = \gamma(\tau_1)$

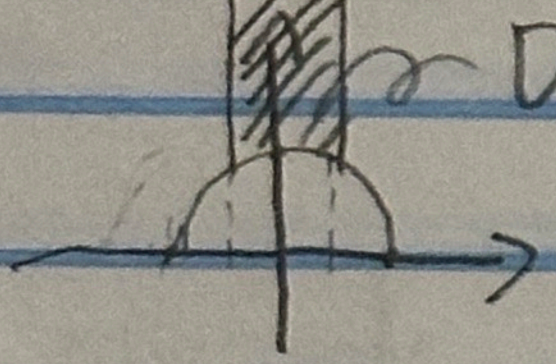
Now, think of it when $\Gamma_0(N), \Gamma_1(N), \Gamma(N)$

Since congruence subgroups of $SL_2(\mathbb{Z})$ has fewer transformations than $SL_2(\mathbb{Z})$, some points that were equ. under $SL_2(\mathbb{Z})$ may become inequivalent under congruence subgroups. ($\underbrace{\tau_1 \sim_{SL_2(\mathbb{Z})} \tau_2}_{\exists \tau_1, \tau_2 \text{ s.t}}$ but $\tau_1 \not\sim_{SL_2(\mathbb{Z})} \tau_2$)

Therefore, the fundamental domain becomes geometrically larger ~~& splits into more pieces~~.

Explanation:

Recall. $D_{SL_2(\mathbb{Z})} = \{ z \in \mathbb{H} : |z| \geq 1, -\frac{1}{2} \leq Re(z) \leq \frac{1}{2} \}$



$D_{SL_2(\mathbb{Z})}$ Just 1 piece.

Intuitive understanding

In $SL_2(\mathbb{Z})$, suppose point τ_1, τ_2, τ_3 are equivalent to each other, & τ_4, τ_5, τ_6 are also equivalent, while τ_1 & τ_4 are inequivalent.

$$\tau_1 \sim \tau_2 \sim \tau_3, \quad \tau_4 \sim \tau_5 \sim \tau_6, \quad \tau_1 \not\sim \tau_4$$

Then, the fundamental domain contains one representative from each class - for example, τ_1 & τ_4.

However, in congruence subgroup, the equivalence relation becomes weaker. It is possible that within the first group $\{\tau_1, \tau_2, \tau_3\}$, one element such as τ_3 is no longer equivalent to the others.

As a result, in the congruence subgroup, the fundamental domain must now include τ_1, τ_4, & τ_3. * ~~Usually, points that newly appear in the fundamental domain (e.g. τ_3) are usually located far apart on the upper half plane~~.

"As the size of a Γ decreses, the fundamental domain becomes larger ~~& splits into more pieces~~."

You could check this with Index Formula

$$[SL_2(\mathbb{Z}) : \Gamma_0(N)] = \frac{\text{area of } D \text{ of } \Gamma_0(N)}{\text{area of } D \text{ of } SL_2(\mathbb{Z})} = N \prod_{\underset{\text{prime \# of } N}{p|N}} \left(1 + \frac{1}{p}\right)$$

$$= \cancel{\text{\# of pieces}}$$

N	Index
2	3
3	4
4	6
6	12
10	

As N gets higher, the size of a $\Gamma_0(N)$ decreases.

Therefore, the f.d becomes larger & ~~splits into more pieces~~.

Now, How does the Riemann Surface change?

Euler Characteristic & area of fundamental domain

$$\text{Area}(D_\Gamma) = 2\pi\left(2g-2+c+\sum_{Q}\left(1-\frac{1}{e_Q}\right)\right)$$

g: # of genus, c: # of cusps

Intuitively, as Area $(D_\Gamma)\uparrow$, # of cusps, genus $\uparrow$,

Riemann Surface becomes topologically more complex.

Finally, how does modular form change?