Lecture Notes 10/15/2025

1. Short Review of Hecke Operators
2. Proof that Hecke Operators are commutative
3. Action on Fourier Coefficients
4. Hecke Eigenforms + Ramanujan's conjectures
5. Petersson Inner Product
6. Hecke Operators are Hermitian
7. Questions ?

1. Review

Modular form q-Expansion: $f(z)$ of weight $2k$ for $\Gamma(1) = SL_2(\mathbb{Z})$ has FS exp:

$$f(z) = \sum_{n=0}^{\infty} c(n) q^n, \text{ where } q = e^{2\pi i z}$$

In particular, we have Ram. τ fn. The discriminant modular form, $\Delta(z)$, is a cusp form of weight 12. Its Fourier coefficients are $\tau(n)$.

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n) q^n$$

C1: p prime, $\gcd(m,n) = 1 \Rightarrow \tau(mn) = \tau(m)\tau(n)$ Multiplicativity

C2: $\tau(p)\tau(p^n) = \tau(p^{n+1}) + p^{11}\tau(p^{n-1})$ for $n \geq 1$ Recurrence relation.

Lattice Def'n of Hecke Operators: Take $\mathcal{L}$ to be the set of all lattices in $\mathbb{C}$

n-th Hecke Operator, $T(n)$ acts on the free abelian gp gen. by $\mathcal{L}$. For $\Lambda \in \mathcal{L}$

$$T(n)[\Lambda] = \sum_{[\Lambda:\Lambda'] = n} [\Lambda']$$

This just means that $T(n)$ acts on the set of lattices in $\mathbb{C}$ by taking a lattice Λ to be the formal sum of all its sublattices of index n.

As a reminder, a lattice in $\mathbb{C}$ is a regularly spaced, infinite grid of points. $\Lambda = \{m\overset{L.I}{\omega_1} + n\overset{L.I \text{ on } \mathbb{R}}{\omega_2} \mid m,n \in \mathbb{Z}\}$ $\omega_1, \omega_2 \in \mathbb{C}$ and are "basis vectors"

<u>Matrix perspective of Hecke Operators</u> Let $M(n) \in GL_2(\mathbb{R})$, $\det(M) = n$ for a modular form f of weight $2K$. Operator is a sum over orbits of integer matrices $M(n)$ under left multiplication by $\Gamma(1)$.

$$(T(n)f)(z) = n^{k-1} \sum_{\alpha_i \in \Gamma(1) \backslash M(n)} (f(_k \alpha_i))(z)$$

where $(f|_k \alpha)(z) = (\det \alpha)^k (cz+d)^{-2k} f\left(\frac{az+b}{cz+d}\right)$ is the slash operator.

<u>Pf that Hecke Ops are commutative (1/2)</u>

1. Recall that $T(n)$ acts on a lattice Λ by producing a formal sum of all its sublattices of index n:

$$T(n)[\Lambda] = \sum_{[\Lambda:\Lambda']=n} [\Lambda']$$

2. Apply operator sequentially to LHS

$$T(m)\,T(n)[\Lambda] = \sum_{[\Lambda:\Lambda']=n} T(m)[\Lambda'] = \sum_{[\Lambda:\Lambda']=n} \sum_{[\Lambda':\Lambda'']=m} [\Lambda'']$$

This is a sum of all lattice chains $\Lambda \supset \Lambda' \supset \Lambda''$ where the indices are m, n respectively. By TR: $[\Lambda:\Lambda''] = [\Lambda:\Lambda'][\Lambda':\Lambda''] = mn$

RHS: $T(mn)$ is by def'n the sum over all sublattices of index mn:

$$T(mn)[\Lambda] = \sum_{[\Lambda:\Lambda'']=mn} [\Lambda'']$$

3. Bijection thanks to CRT. We want to show (1-1) $\exists!$ interm. sublattice Λ' in the chain for any sublattice $\Lambda'' \subset \Lambda$ of index mn.

order $(\Lambda/\Lambda'') = mn$ finite abel. gp

$\gcd(m,n) = 1 \Rightarrow$ by CRT, Λ/Λ'' has a unique subgp of order m

By gp theory correspondence, this unique subgp of Λ/Λ'' corresponds to a unique intermediate lattice Λ' s.t $\Lambda \supset \Lambda' \supset \Lambda''$ and $[\Lambda' : \Lambda''] = m$.

Therefore, it follows that $[\Lambda : \Lambda''] = [\Lambda : \Lambda'][\Lambda' : \Lambda''] = mn$

$\Rightarrow [\Lambda : \Lambda'] = [\Lambda : \Lambda''] / [\Lambda' : \Lambda''] = mn/m = n \Rightarrow$ 1-1.

$T(m)T(n) = T(mn)$. Proved multiplicativity, from which comm. imm. follows.
(for $\gcd(m,n) = 1$)

3. Action on fourier coefficients

Modular form q expansion $f(z)$ weight $2k$ for $\Gamma(1) = SL_2(\mathbb{Z})$

$$f(z) = \sum_{m=0}^{\infty} c(m) q^m \in M_{2k}(\Gamma(1)), \text{ where } q = e^{2\pi i z}$$

We use the standard set of representatives for the orbits $\Gamma(1) \backslash M(n)$,

$$\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \quad ad = n, a \geq 1 \text{ and } 0 \leq b < d$$

The action of $T(n)$ is: by the slash

$$(T(n)f)(z) = n^{k-1} \sum_{ad=n, a\geq 1} \sum_{b=0}^{d-1} \left(f|_k \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}\right)(z) = n^{k-1} \sum_{ad=n,a\geq 1} \sum_{b=0}^{d-1} n^k d^{-2k} f\left(\frac{az+b}{d}\right)$$

Now we substitute the fourier series for f:

$$= n^{2k-1} \sum_{ad=n,a\geq 1} d^{-2k} \sum_{b=0}^{d-1} \sum_{m=0}^{\infty} c(m) e^{2\pi i m \left(\frac{az+b}{d}\right)} = n^{2k-1} \sum_{ad=n,a\geq 1} d^{-2k} \sum_{m=0}^{\infty} c(m) e^{2\pi i m \frac{az}{d}} \left(\sum_{b=0}^{d-1} e^{2\pi i m \frac{b}{d}}\right)$$

All we did was split up the exponents... Why? Roots of unity

Now notice the sum over b is a sum of roots of unity. Let $e^{2\pi i m/d} = \zeta$.

The sum becomes $\sum_{b=0}^{d-1} \zeta^b$.

Notice: If $\zeta = 1$ which happens iff $d|m$, sum is $\overbrace{1+1+\ldots+1}^{d \text{ times}} = d$

Else, sum is $\frac{\zeta^d - 1}{\zeta - 1} = \frac{(e^{2\pi i m/d})^d - 1}{\zeta - 1} = \frac{1-1}{\zeta-1} = \frac{0}{\zeta-1} = 0$ by geom. series

nonzero iff m mult of $d \Rightarrow m = jd$

$$(T(n)f)(z) = n^{2k-1} \sum_{ad=n, a\geq 1} d^{-2k} \cdot d \sum_{j=0}^{\infty} c(jd) e^{2\pi i (jd)\frac{az}{d}}$$

$$= \sum_{ad=n, a\geq 1} (ad)^{2k-1} d^{-2k+1} \sum_{j=0}^{\infty} c(jd) e^{2\pi i j a z} = \sum_{ad=n, a\geq 1} a^{2k-1} \sum_{j=0}^{\infty} c(jd)(q^a)^j$$

Let the Fourier Expansion of $(T(n)f)(z)$ be $\sum_{m=0}^{\infty} \gamma_m q^m$. Want to find γ_m. This coefficient is the sum of all terms where $aj = m$. Why? Look exp.

So, $\gamma_m = \sum_{a|(m,n)} a^{2k-1} c\left(\frac{m}{a} d\right) = \sum_{a|(m,n)} a^{2k-1} c\left(\frac{mn}{a^2}\right)$ since $d = \frac{n}{a}$.

Now, notice the simplicities when p is prime. mth coeff of $T(p)f$ is:

$$\gamma_m = c(mp) + p^{2k-1} c(m/p)$$

Since Case 1: $p \nmid m \Rightarrow \gcd(m,p) = 1$ so $d=1, n=p \Rightarrow c(mp)$

Case 2: $p | m \Rightarrow \gcd(m,p) = p \Rightarrow 1^{2k-1} c\left(\frac{mp}{1^2}\right) + p^{2k-1} c(m/p)$

So if p is prime, the mth coefficient of $(T(p)f)(z)$ is $c(mp)$ $+ p^{2k-1} c(m/p)$ where we define $c(x) = 0$ if x not an integer.

4. Hecke Eigen forms + Ramanujan's conjectures

In L.A, the most natural basis for a V.S w/ a linear operator is a basis of eigenvectors. We want this same idea in mod forms + Hecke.

<u>Def'n</u>: A non-zero modular form $f \in M_{2k}(\Gamma(1))$ is a Hecke eigenform if it is an eigenvector for every Hecke operator $T(n)$ $(n \geq 1)$. That is, for each n, $\exists\ \lambda_n \in \mathbb{C}$ s.t $T(n)f = \lambda_n f$

λ_n is the e.v corresponding to $T(n)$.

This is a $\boxed{\text{VERY STRONG}}$ condition. A priori, there's no reason for a form to be an ev. for more than 1 operator, let alone all of them!

Why? Question to class. A: $T(m)T(n) = T(n)T(m)$ commutativity.

<u>Thm</u>: Eigenvalues are Fourier Coefficients

Let $f(z) = \sum_{m=0}^{\infty} c(m)q^m$ be a Hecke eigenform. If f is normalized s.t $c(1) = 1$, then its eigenvalues are precisely its Four. coef

$$\lambda_n = c(n) \quad \forall\ n \geq 1$$

<u>Pf</u>: Let $T(n)f = \lambda_n f$. Note that this is an equality of modular forms. For them to be equal, all of their Fourier coefficients equal. com pre. coefficient of q^1 on both sides

RHS $(\lambda_n f)$: F.E of $\lambda_n f$ is $\lambda_n \sum c(m)q^m = \sum (\lambda_n c(m))q^m$.

The coefficient of q^1 is $\lambda_n c(1)$. Since f is normalized, $c(1) = 1 \Rightarrow$ coefficient is just simply λ_n.

LHS: Use our general formula for the m-th coefficient of $T(n)f$, which we called $\gamma(m)$ and set $m=1$. Then,

$$\gamma(1) = \sum_{d \mid \gcd(1,n)} d^{2k-1} c\left(\frac{1 \cdot n}{d^2}\right).$$ But $\gcd(1,n)=1$ always $\Rightarrow d=1$ only pos. divisor

$$\gamma(1) = 1^{2k-1} c\left(\frac{1\cdot n}{1^2}\right) = c(n).$$ Equating the coefficients from both sides gives $\lambda_n = C(n)$.

Quick Note on Normalization:

For $f(z) = \sum_{n=0}^{\infty} c(n) q^n$ a Hecke eigenform, if $c(1) \neq 0$,

$$f_{norm}(z) = \frac{1}{c(1)} \sum_{n=0}^{\infty} c(n) q^n \Rightarrow C_{norm}(1) = \frac{c(1)}{c(1)} = 1.$$

Ramanujan's conjectures

Lemma: The discriminant $\Delta(z) = \sum_{n=1}^{\infty} \tau(n) q^n$ is a Hecke Eigenform

1. Complex v. space of cusp forms of weight 12 for $\Gamma(1)$, S_{12}, $\dim(S_{12}) = 1$
2. Since $\Delta(z)$ is a nonzero element of S_{12}, it is a basis vector for S_{12}.
3. $T(n): S_{12} \to S_{12}$, Hecke operators linear
4. If V is 1D with b. vector $\vec{v}$: $L: V \to V$ sends $\vec{v}$ to $\lambda \vec{v}$.

$\Rightarrow$ for any $T(n)$, it must map $\Delta(z)$ to a sc. mult $\Rightarrow n \geq 1\ \exists \lambda_n \in \mathbb{C}$ s.t
$T(n)\Delta = \lambda_n \Delta$. Notice that $\tau(1) = 1 \Rightarrow$ normalized.

$\Rightarrow \lambda_n = \tau(n)\ \forall n \geq 1$ by our theorem

Multiplicativity: $T(m)(T(n)\Delta) = T(mn)\Delta \Rightarrow T(m)(\lambda_n \Delta) = \lambda_{mn}\Delta$
$\Rightarrow \lambda_n (T(m)\Delta) = \lambda_{mn}\Delta$ and then $\lambda_n \lambda_m \Delta = \lambda_{mn}\Delta$. Since $\Delta \neq 0$ F.
$\lambda_m \lambda_n = \lambda_{mn}$ Substitute $\lambda_n = \tau(n)$

$$\Rightarrow \tau(m)\tau(n) = \tau(mn)$$

Now notice that $d\left(-\frac{1}{z}\right) = \frac{dz}{z^2}$ and $d\left(-\frac{1}{\bar{z}}\right) = \frac{d\bar{z}}{\bar{z}^2}$ and

$\mathrm{Im}\left(-\frac{1}{z}\right) = \frac{\mathrm{Im}(z)}{z\bar{z}}$. Let's show this:

$$d\left(-\frac{1}{z}\right) = \frac{d}{dz}\left(-z^{-1}\right)dz = (-1)(-1)\,z^{-2}\,dz = \frac{dz}{z^2}$$

Similarly, for the conjugate variable, $d(-1/\bar{z}) = \frac{d\bar{z}}{\bar{z}^2}$.

$-\frac{1}{z} = \frac{-1}{x+iy} = \frac{-x+iy}{x^2+y^2} \Rightarrow \mathrm{Im}\left(-\frac{1}{z}\right) = \frac{y}{x^2+y^2}$. See $\mathrm{Im}(z) = y$ &

$|z|^2 = z\bar{z} = x^2+y^2 \Rightarrow \mathrm{Im}(-1/z) = \frac{\mathrm{Im}(z)}{|z|^2}$

Therefore, through some calculations, you can show $\frac{dz\,d\bar{z}}{(\mathrm{Im}\,z)^2}$ is invariant under $SL_2(\mathbb{R})$

If f, g weight k s.t $f(z)(dz)^{k/2}$ and $g(\bar{z})(d\bar{z})^{k/2}$ both invariant,

find that $f(z)\bar{g}(z)(\mathrm{Im}\,z)^k$ is invariant because the transformation of the imaginary part of z cancels out with the transformations of dz and $d\bar{z}$.

$$(f, g) = \int_F f(z)\overline{g(z)}\,(\mathrm{Im}(z))^k \frac{dx\,dy}{\mathrm{Im}(z)^2}$$, both parts are invariant.

Well-definedness follows from invariance, since both parts are invariant under the action of $SL_2(\mathbb{Z})$

L.A Review: find $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ that satisfies

1) $\langle u + \lambda v, w \rangle = \langle u, w \rangle + \lambda \langle v, w \rangle$

2) $\langle v, v \rangle = \overline{\langle v, u \rangle}$ Hermitian Inner product

3) $\langle u, u \rangle \geq 0$ with $\langle u, u \rangle = 0 \Leftrightarrow u = 0$

Recurrence relation: $T(p)T(p^n) = T(p^{n+1}) + p^{11}T(p^{n-1})$. weight $2k=12$

Replace operators w/ corresponding eigenvalues gives the identity:

$\lambda_p \lambda_{p^n} = \lambda_{p^{n+1}} + p^{11}\lambda_{p^{n-1}}$. Substitute $\lambda_n = \tau(n)$, we get:

$\tau(p)\tau(p^n) = \tau(p^{n+1}) + p^{11}\tau(p^{n-1})$ and we are done.

Petersson Inner Product

Motivation: A way to measure lengths and angles in space of mod forms.

~~Since~~ On $\mathbb{H}$, there is a measure $\frac{dx\,dy}{y^2}$ ⇐ invariant under $SL_2(\mathbb{R})$

Natural to integrate w/ respect

- f, g mod forms of weight k for some $\Gamma \subset \Gamma(1)$

$f \cdot g$ and $f \cdot \bar{g}$ NOT invariant under $SL_2\mathbb{Z}$.

↳ no differential forms to cancel

$\int_{F.D} f \cdot \bar{g} \frac{dx\,dy}{y^2}$ completely undefined. $z = x+iy \in \mathbb{C}$

$f(z)(dz)^{k/2}$ is invariant, and so is $\overline{g(z)(d\bar{z})^{k/2}}$.

↳ $z' = \gamma z = \frac{az+b}{cz+d}$ where $\gamma \in SL_2(\mathbb{Z})$.

1. $f(z)(dz)^{k/2}$ ⇒

- Transforms as $f(z') = (cz+d)^k f(z)$ Modular form

- differential dz: $dz' = \frac{d}{dz}\left(\frac{az+b}{cz+d}\right)dz = \frac{ad-bc}{(cz+d)^2}dz$. Since $ad-bc=1$ in $SL_2(\mathbb{Z})$ ⇒ $dz' = \frac{1}{(cz+d)^2}dz$.

$f(z')(dz')^{k/2} = ((cz+d)^k f(z)) \cdot \left(\frac{1}{(cz+d)^2}dz\right)^{k/2} = f(z)(dz)^{k/2}$

Analogous reasoning for the case of $\overline{g(z)(d\bar{z})^{k/2}}$

The Petersson inner product satisfies there, so $S_k(\Gamma(1))$ is a finite-dimensional Hilbert space.

↳ complete inner product space

Proposition: Hecke Operators are Hermitian on space cusp forms.

i.e $\langle T(n)f, g\rangle = \langle f, T(n)g\rangle$. "moved from one side to other".

Won't prove. Unfold integral over f.d and use matrix defn of Hecke.

Thm: Spectral Theorem

For any commuting family of Hermitian operators on a finite dimensional Hilbert space, $\exists$ orthogonal basis of sim. eigenvectors for all operators in that family.

Proposition: Eigenvalues of Eigenforms are real

pf. Let f be a normalized Hecke eigenform $\Rightarrow \lambda_n = c(n)$.

1. $T(n)f = \lambda_n f$. $\langle T(n)f, f\rangle = \langle \lambda_n f, f\rangle = \lambda_n \langle f, f\rangle$

2. By Hermitian property: $\langle T(n)f, f\rangle = \langle f, T(n)f\rangle$

3. $\langle f, \lambda_n f\rangle$. By conjugate linearity of IP, $\overline{\lambda_n}\langle f, f\rangle$

4. $\lambda_n \langle f, f\rangle = \overline{\lambda_n}\langle f, f\rangle$.

5. Since f is not the zero form, $\langle f, f\rangle \neq 0 \Rightarrow \lambda_n = \overline{\lambda_n}$ $\square$

Hence, the Fourier coefficients of a normalized Hecke eigenform $\in \mathbb{R}$

Proposition: $S_k(\Gamma(1))$ has a basis $\{f_1, f_2, \dots, f_d\}$ that are:

1) Simultaneous Eigenform: Each f_i an eigenform $\forall$ $T(n)$

2) Orthogonal: $\langle f_i, f_j\rangle = 0$ if $i \neq j$

Tangent on modular functions:

Measure (hyp): calculate area in the upper half plane

Measure $\frac{dx\,dy}{y^2}$ on $\mathbb{H}$ which is invariant under $SL_2(\mathbb{R})$

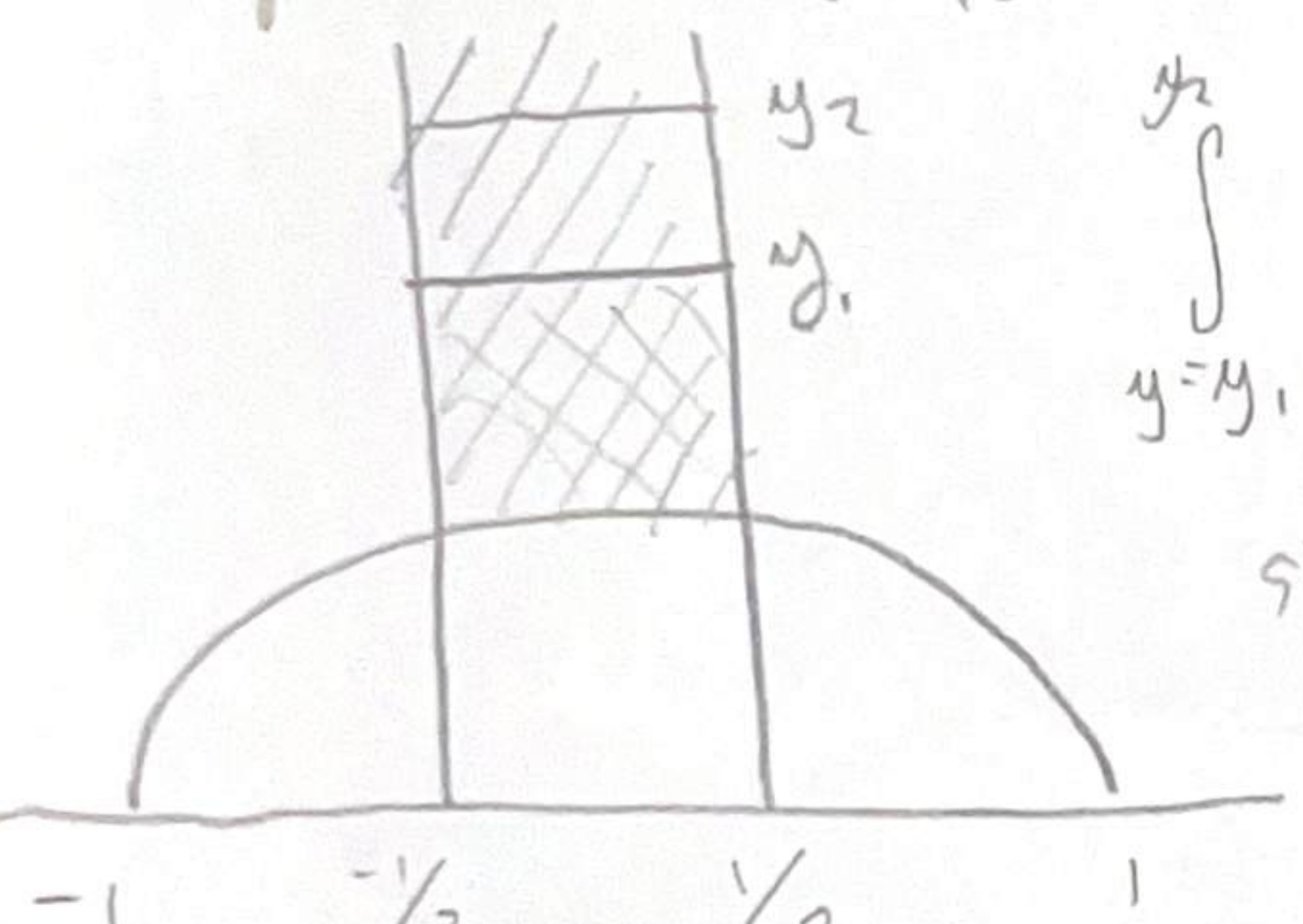
Let f, g mod functions $\langle f,g\rangle = \int fg \frac{dx\,dy}{y^2}$ F.D. of $SL_2\mathbb{Z}$.

Makes sense, but it diverges $\longrightarrow$ f is some q function

It's term q^{-n}, $q = e^{2\pi i z}$ $\quad |q^{-n}| = e^{2\pi n y}$ - very big

Regularize

Works but depends on cutoff



$$\int_{y=y_1}^{y_2}\int_{x=-1/2}^{1/2} (q^{-n} + \dots +) = 0 \text{ if } n \neq 0$$

since $q^{-n} = e^{2\pi i(x+iy)n}$, integral x $-\frac12$ to $\frac12$ is 0.

$$\lim_{y\to\infty} \int_{\substack{\text{F.D.} \\ \text{cutoff at } y}} f \cdot g \frac{dx\,dy}{y^2}.$$ Limit wd if f, g holomorphic mod fnctns.

$\frac{1}{2}(w+2b) = w + \frac{b}{2}$

May or may not have poles

$s_n := \sup_{k\geq n} x_k$ and $t_n := \inf_{k\geq n} x_k$ Then $s_{n+1} \leq s_n$ and $t_{n+1} \geq t_n$ and

$\limsup x_n = \inf_n s_n \quad \liminf x_n = \sup_n t_n$

$\Rightarrow$ If $x_n \to L$ then $\limsup x_n = \liminf x_n = L$

given $\varepsilon > 0$ $\exists N$ with $L - \varepsilon < x_n < L + \varepsilon$ for all $n \geq N$. Therefore, for $n \geq N$, $t_n \geq L - \varepsilon$, $s_n \leq L + \varepsilon$. Take $\sup_n t_n$ and $\inf_n s_n$: $\liminf x_n \geq L - \varepsilon$, $\limsup x_n \leq L + \varepsilon$.

Let $\varepsilon \downarrow 0$: $\liminf x_n \geq L \geq \limsup x_n$. $\Rightarrow$ equality since $\liminf \leq \limsup$

$\Leftarrow$ If $\limsup x_n = \liminf x_n =: L$, then $x_n \to L$

Fix $\varepsilon > 0$. Since $L = \inf_n s_n$, take ~~N~~ N_1 w/ $s_{N_1} < L + \varepsilon$. Since $L = \sup_n t_n$, pick N_2 w/ $t_{N_2} > L - \varepsilon$. Let $L = \max\{N_1, N_2\}$. $\forall n \geq N$, $L - \varepsilon < t_n \leq x_n \leq s_n < L + \varepsilon$

so $|x_n - L| < \varepsilon \Rightarrow x_n \to L$