Hecke Operators I

Sebastian Pallais-Aks

October 13th, 2025

Outline

Fourier Expansions

Modular Form q-Expansion

A modular form $f(z)$ of weight $2k$ for $\Gamma(1) = SL_2(\mathbb{Z})$ has a Fourier series expansion:

$$f(z) = \sum_{n=0}^{\infty} c(n) q^n, \quad \text{where } q = e^{2\pi i z}$$

Example: The Ramanujan τ-Function

The discriminant modular form, $\Delta(z)$, is a cusp form of weight 12. Its Fourier coefficients are the τ-function, $\tau(n)$.

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n) q^n$$

Ramanujan's Conjectures

Conjectures for $\tau(n)$

For a prime p and coprime integers m, n:

1. **Multiplicativity:**
   $\tau(mn) = \tau(m)\tau(n)$
2. **Recurrence Relation:**
   $\tau(p)\tau(p^n) = \tau(p^{n+1}) + p^{11}\tau(p^{n-1})$ for $n \geq 1$.

Two Perspectives: Lattice Definition

Lattice Definition

- Let $\mathcal{L}$ be the set of all lattices in $\mathbb{C}$.
- The n-th **Hecke Operator**, $T(n)$, acts on the free abelian group generated by $\mathcal{L}$. For a lattice $\Lambda \in \mathcal{L}$:

$$T(n)[\Lambda] = \sum_{[\Lambda:\Lambda']=n} [\Lambda']$$

- **Action on Functions:** For a function $F : \mathcal{L} \to \mathbb{C}$:

$$(T(n)F)(\Lambda) = \sum_{[\Lambda:\Lambda']=n} F(\Lambda')$$

Geometric Motivation for Double Cosets

The Problem with a Simple Action

We want to define an action of a matrix $\alpha \in GL_2(\mathbb{R})^+$ on the quotient space $\Gamma\backslash\mathbb{H}$.

- A map defined as $\Gamma z \mapsto \Gamma\alpha z$ is **not well-defined**.
- This is because Γ is not a normal subgroup of $GL_2(\mathbb{R})^+$, so in general $\alpha^{-1}\Gamma\alpha \neq \Gamma$. The orbit $\Gamma\alpha z$ depends on the choice of representative z.

Geometric Motivation for Double Cosets

The Double Coset Solution

- Instead of a single matrix α, we consider the entire **double coset** $\Gamma\alpha\Gamma$.
- This double coset can be decomposed into a finite, disjoint union of right cosets:

$$\Gamma\alpha\Gamma = \bigcup_i \Gamma\alpha_i$$

- This defines a "many-valued map" or a **correspondence** on the quotient space:

$$\Gamma z \mapsto \{\Gamma\alpha_1 z, \Gamma\alpha_2 z, \dots\}$$

The Hecke Correspondence

The Action as Pullback and Trace

The correspondence can be visualized with a diagram where $\Gamma_\alpha = \Gamma \cap \alpha^{-1}\Gamma\alpha$ and $\Gamma = \bigcup \Gamma_\alpha \cdot \beta_i$.

$$\begin{tikzcd}
& \Gamma_\alpha \backslash \mathbb{H} \arrow[dl] \arrow[dr, "\alpha"] & \\
\Gamma \backslash \mathbb{H} & & \Gamma \backslash \mathbb{H}.
\end{tikzcd}$$

The action of the Hecke operator on a modular function f is interpreted as a pullback $(f \circ \alpha)$ followed by a trace (summing over the cosets).

The Hecke Algebra: Formal Definition

As a Free $\mathbb{Z}$-Module

The Hecke Algebra, denoted $\mathcal{H}(\Gamma, \Delta)$, is the free $\mathbb{Z}$-module generated by the double cosets $\Gamma\alpha\Gamma$ for $\alpha \in \Delta$.

- An element of the algebra is a formal finite sum with integer coefficients:

$$\sum n_\alpha \Gamma\alpha\Gamma, \quad n_\alpha \in \mathbb{Z}$$

The Hecke Algebra

Multiplication in the Algebra

We define a multiplication on $\mathcal{H}(\Gamma, \Delta)$:

- First, decompose each double coset into a finite union of disjoint right cosets:

$$\Gamma\alpha\Gamma = \bigcup_i \Gamma\alpha_i \quad \text{and} \quad \Gamma\alpha\Gamma = \bigcup_j \Gamma\beta_j$$

  Then

$$\Gamma\alpha\Gamma \cdot \Gamma\beta\Gamma = \Gamma\alpha\Gamma\beta\Gamma = \bigcup_{i,j} \Gamma\alpha_i\beta_j$$

- The product is then a formal sum over the resulting double cosets $[\gamma]$:

$$[\alpha] \cdot [\beta] = \sum_{\gamma \in \Delta} c^{\gamma}_{\alpha,\beta}[\gamma]$$

- The integer coefficients $c^{\gamma}_{\alpha,\beta}$, are defined as the number of pairs (i, j) with $\Gamma\alpha_i\beta_j = \Gamma_\gamma$.

Two Perspectives: Matrix (Double Coset) Definition

Matrix Definition

- Let $M(n)$ be the set of 2×2 integer matrices with determinant n.
- **Slash Operator:** The action of $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R})^+$ on a function $f(z)$ of weight $2k$:

$$(f|_k\alpha)(z) = (\det \alpha)^k (cz+d)^{-2k} f\left(\frac{az+b}{cz+d}\right)$$

- **Hecke Operator:** A sum over the orbits of $\Gamma(1)\backslash M(n)$:

$$(T(n)f)(z) = n^{k-1} \sum_{\alpha_i \in \Gamma(1)\backslash M(n)} (f|_k\alpha_i)(z)$$

A standard set of representatives for the orbits is $\left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mid ad = n, a \geq 1, 0 \leq b < d \right\}$.

Hecke Operator Formula

Using either definition, we can also define Hecke operators on modular forms: $T(n)f(z)$ is the function on $\mathbb{H}$ associated with $n^{2k+1} \cdot T(n) \cdot F$. n^{2k-1} is inserted to ensure integer coefficients. Thus:

$$T(n)f(z) = n^{2k-1} \cdot \sum d^{-2k} f(\frac{az+b}{d})$$

Operator Identities

The **homothety operator** R_λ maps a lattice Λ to $\lambda\Lambda$.

1. $R_\lambda R_\mu = R_{\lambda\mu}$ (trivial)
2. $R_\lambda T(n) = T(n) R_\lambda$ (trivial)
3. If $\gcd(m, n) = 1$, then $T(m)T(n) = T(mn)$
4. If p is prime, $T(p)T(p^n) = T(p^{n+1}) + pR_p T(p^{n-1})$

Proof Sketch: Multiplicativity

Identity 3: $T(m)T(n) = T(mn)$ for $\gcd(m, n) = 1$

- The LHS, $T(m)T(n)[\Lambda]$, sums over chains $\Lambda \supset \Lambda' \supset \Lambda''$ with indices $[\Lambda : \Lambda'] = n$ and $[\Lambda' : \Lambda''] = m$.
- The total index is $[\Lambda : \Lambda''] = mn$.The quotient group Λ/Λ'' has order mn.
- By the Chinese Remainder Theorem, since m, n are coprime, Λ/Λ'' has a **unique** subgroup of order m.
- This unique subgroup corresponds to a unique intermediate lattice Λ'.
- This establishes a one-to-one correspondence between the terms on both sides.

Proof Sketch: Prime Power Recurrence

Identity 4: $T(p)T(p^n) = T(p^{n+1}) + pR_pT(p^{n-1})$

Fix a sublattice $\Lambda'' \subset \Lambda$ of index p^{n+1} and compare its coefficient on each side.

- **Case 1:** $\Lambda'' \not\subset p\Lambda$
  - *LHS:* The coefficient is the number of intermediate lattices Λ' of index p. Such a Λ' must correspond to the same line in $\Lambda/p\Lambda$ as Λ'', so there is only **one**.
  - *RHS:* $T(p^{n+1})$ contributes 1. The R_p term contributes 0.
  - Result: $1 = 1 + 0$.

- **Case 2:** $\Lambda'' \subset p\Lambda$
  - *LHS:* Any lattice Λ' of index p contains $p\Lambda$ and thus Λ''. There are $\mathbf{p+1}$ such lattices (lines in an $\mathbb{F}_p$-plane).
  - *RHS:* $T(p^{n+1})$ contributes 1. The $pR_pT(p^{n-1})$ term contributes p.
  - Result: $p + 1 = 1 + p$.