§4. Modular Forms: Expansions at Infinity and the Discriminant Δ

All modular forms are for the full modular group $SL_2(\mathbb{Z})$.
We index Eisenstein series by weight: G_{2k}, E_{2k}.

4.0 Recap

(1) Bernoulli numbers and $\zeta(2k)$. The Bernoulli numbers B_{2k} are defined by

$$\frac{x}{e^x - 1} = 1 - \frac{x}{2} + \sum_{k\geq 1} \frac{B_{2k}}{(2k)!} x^{2k},$$

and yield the special values

$$\zeta(2k) = (-1)^{k+1} \frac{B_{2k}(2\pi)^{2k}}{2(2k)!}.$$

These constants appear in the normalization of Eisenstein series.

(2) Eisenstein series G_{2k} and E_{2k}. For integers $k \geq 2$,

$$G_{2k}(z) = \sum_{(m,n)\neq(0,0)} \frac{1}{(m+nz)^{2k}}, \qquad q = e^{2\pi i z}.$$

One obtains the q-expansion

$$G_{2k}(z) = 2\zeta(2k) + \frac{2(2\pi i)^{2k}}{(2k-1)!} \sum_{n\geq 1} \sigma_{2k-1}(n) q^n,$$

where $\sigma_{2k-1}(n) = \sum_{d|n} d^{\,2k-1}$. The normalized form is

$$E_{2k}(z) = 1 - \frac{4k}{B_{2k}} \sum_{n\geq 1} \sigma_{2k-1}(n) q^n,$$

For the full modular group one has the structure theorem

$$\mathcal{M} = \bigoplus_{k\geq 0} \mathcal{M}_{2k} \cong \mathbb{C}[E_4, E_6],$$

so every holomorphic modular form can be written as a polynomial in E_4 and E_6. In particular $E_8 = E_4^2$, $E_{10} = E_4 E_6$.

4.1 Definition and Initial Expansion of Δ

Set $q = e^{2\pi i z}$, $z \in \mathbb{H}$. We will show that

$$\Delta = (2\pi)^{12}\, q \prod_{n=1}^{\infty} (1 - q^n)^{24}.$$

$$\begin{aligned} \text{Recall that} \quad \Delta &= g_4^3 - 27 g_6^2 = (2\pi)^{12}\, 2^{-6}\, 3^{-3}\, (E_4^3 - E_6^2) \\ &=_? (2\pi)^{12} \big(q - 24q^2 + 252q^3 - 1472q^4 + \cdots \big). \end{aligned}$$

whose coefficients define the Ramanujan function $\tau(n)$. The space $S_{12}(SL_2(\mathbb{Z}))$ is one-dimensional, so every weight-12 cusp form is proportional to Δ.

(Skip) How we derived $\Delta = g_4^3 - 27\, g_6^2 = (2\pi)^{12}\, 2^{-6}\, 3^{-3}\, (E_4^3 - E_6^2)$.

(1) Classical relations.

$$g_4 = 60\, G_4, \qquad g_6 = 140\, G_6,$$

$$G_{2k}(z) = 2\,\zeta(2k)\, E_{2k}(z) \quad \Rightarrow \quad \begin{cases} G_4 = \dfrac{\pi^4}{45}\, E_4, \\ G_6 = \dfrac{2\pi^6}{945}\, E_6. \end{cases}$$

(2) Express g_4, g_6 via E_4, E_6.

$$\begin{aligned} g_4 &= 60 \cdot \frac{\pi^4}{45}\, E_4 = \frac{4}{3}\, \pi^4\, E_4, \\ g_6 &= 140 \cdot \frac{2\pi^6}{945}\, E_6 = \frac{280}{945}\, \pi^6\, E_6 = \frac{8}{27}\, \pi^6\, E_6. \end{aligned}$$

(3) Plug into $g_4^3 - 27 g_6^2$.

$$\begin{aligned} g_4^3 &= \left(\tfrac{4}{3} \pi^4 E_4 \right)^3 = \frac{64}{27}\, \pi^{12}\, E_4^3, \\ g_6^2 &= \left(\tfrac{8}{27} \pi^6 E_6 \right)^2 = \frac{64}{729}\, \pi^{12}\, E_6^2, \\ g_4^3 - 27\, g_6^2 &= \frac{64}{27}\, \pi^{12}\, E_4^3 - 27 \cdot \frac{64}{729}\, \pi^{12}\, E_6^2 = \frac{64}{27}\, \pi^{12}\, (E_4^3 - E_6^2). \end{aligned}$$

(4) Rewrite the constant.

$$\frac{64}{27}\, \pi^{12} = \frac{2^6}{3^3}\, \pi^{12} = (2\pi)^{12}\, 2^{-6}\, 3^{-3}.$$

Therefore

$$\boxed{\Delta = g_4^3 - 27\, g_6^2 = (2\pi)^{12}\, 2^{-6}\, 3^{-3}\, (E_4^3 - E_6^2)}\,.$$

4.2 Proof of Modularity

Let $F(z) = q\prod_{n=1}^{\infty}(1-q^n)^{24}$. It suffices to show

$$F(-1/z) = z^{12}F(z).$$

Introduce the double series

$$G_2(z) = \sum_{n,m}{}' \frac{1}{(m+nz)^2}, \qquad G(z) = \sum_{m,n}{}' \frac{1}{(m+nz)^2},$$

(with $'$ meaning $(m,n) \neq (0,0)$),

Note on proof This formula is proved in the most natural way by using elliptic functions. For more details, A. HURWITZ, Math. Werke, Bd. I, pp. 578-595.

More notes on proof To prove that F and Δ are proportional, it suffices to show that F is a modular form of weight 12; indeed, the fact that the expansion of G has constant term zero will show that F is a cusp form and we know that the space of cusp forms of weight 12 is of dimension 1.

Back to the "proof". define the auxiliary series

$$H_2(z) = \sum_{n,m}{}' \frac{1}{(m-1+nz)(m+nz)}, \qquad H(z) = \sum_{m,n}{}' \frac{1}{(m-1+nz)(m+nz)},$$

where $\sum'$ means the stated pairs $(m,n) \neq (0,0), (0,1)$. Using

$$\frac{1}{(m-1+nz)(m+nz)} = \frac{1}{m-1+nz} - \frac{1}{m+nz},$$

One finds that they converge, and that

$$H_2 = 2, \qquad H = 2 - \frac{2\pi i}{z}.$$

See if time for proof.

Moreover, the double series with general term

$$\frac{1}{(m-1+nz)(m+nz)} - \frac{1}{(m+nz)^2} = \frac{1}{(m+nz)^2(m-1+nz)}$$

is absolutely summable. This shows that $G_2 - H_2$ and $G - H$ coincide $(G-H) = (G_2 - H_2)$, thus that the series G and G_2 converge (with the indicated order of summation) and that

$$G_2(z) - G(z) = H_2(z) - H(z) = \frac{2\pi i}{z}.$$

From this and the transformation $G_2(-1/z) = z^2 G(z)$ one deduces the key identity

$$G_2(-1/z) = z^2 G_2(z) - 2\pi i\, z. \tag{45}$$

On the other hand, with the computation from Jasper, we have

$$G_2(z) = \frac{\pi^2}{3} - 8\pi^2 \sum_{n=1}^{\infty} \sigma_1(n) q^n. \tag{46}$$

Now, for $F(z) = q\prod_{n\geq 1}(1-q^n)^{24}$, its logarithmic differential is

$$\frac{dF}{F} = \frac{dq}{q}\left(1 - 24\sum_{n,m=1}^{\infty} n\, q^{nm}\right) = \frac{dq}{q}\left(1 - 24\sum_{n=1}^{\infty}\sigma_1(n)\, q^n\right). \tag{47}$$

$$= (2\pi i dz)\left(1 - 24\sum_{n=1}^{\infty}\sigma_1(n)\, q^n\right)$$

Comparing (46) and (47) gives

$$\frac{dF}{F} = \frac{6i}{\pi}\, G_2(z)\, dz. \tag{48}$$

Combining (45) (namely $G_2(-1/z) = z^2 G_2(z) - 2\pi i\, z$) with (48),

$$\frac{dF(-1/z)}{F(-1/z)} = \frac{6i}{\pi}\, G_2(-1/z)\,\frac{dz}{z^2} = \frac{6i}{\pi}\,\frac{dz}{z^2}\big(z^2 G_2(z) - 2\pi i\, z\big) = \frac{dF(z)}{F(z)} + 12\,\frac{dz}{z}. \tag{49}$$

Thus $F(-1/z)$ and $z^{12}F(z)$ have the same logarithmic differential, so $F(-1/z) = k\, z^{12}F(z)$ for some constant k. Evaluating at $z = i$, we have $z^{12} = 1,\ -1/z = z$ and $F(z) \neq 0$, so $k = 1$, proving $F(-1/z) = z^{12}F(z)$.

4.3 Ramanujan's τ-Function

We denote by $\tau(n)$ the nth coefficient of the cusp form $F(z) = (2\pi)^{-12}\Delta(z)$. Thus

$$\sum_{n=1}^{\infty} \tau(n) q^n = q \prod_{n=1}^{\infty} (1-q^n)^{24}. \tag{50}$$

The function $n \mapsto \tau(n)$ is called the *Ramanujan function.*

Numerical table.

$$\begin{aligned}
&\tau(1) = 1, \quad \tau(2) = -24, \quad \tau(3) = 252, \quad \tau(4) = -1472, \quad \tau(5) = 4830,\\
&\tau(6) = -6048, \quad \tau(7) = -16744, \quad \tau(8) = 84480, \quad \tau(9) = -113643,\\
&\tau(10) = -115920, \quad \tau(11) = 534612, \quad \tau(12) = -370944.
\end{aligned}$$

Properties of $\tau(n)$.

$$\tau(n) = O(n^6), \tag{51}$$

because Δ has weight 12 (sharpened by Deligne to $\tau(n) = O(n^{11/2+\varepsilon})$ for every $\varepsilon > 0$).

$$\tau(nm) = \tau(n)\tau(m) \quad \text{if } \gcd(n,m) = 1, \tag{52}$$

$$\tau(p^{r+1}) = \tau(p)\tau(p^r) - p^{11}\tau(p^{r-1}) \quad (p \text{ prime}, \ r \geq 1). \tag{53}$$

Define the Dirichlet series

$$L_\tau(s) = \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s},$$

then

$$L_\tau(s) = \prod_p \frac{1}{1 - \tau(p)p^{-s} + p^{11-2s}}. \tag{54}$$

By a theorem of Hecke , $L_\tau(s)$ extends to an entire function in $\mathbb{C}$ and $(2\pi)^{-s}\Gamma(s)L_\tau(s)$ is invariant under $s \mapsto 12 - s$.

Congruences. $\tau(n)$ has various congruences modulo $2^12, 3^6, 5^3, 7, 23, 691$,

$$\tau(n) \equiv n^2\sigma_7(n) \pmod{3^3}, \tag{55}$$

$$\tau(n) \equiv n\,\sigma_3(n) \pmod{7}, \tag{56}$$

$$\tau(n) \equiv \sigma_{11}(n) \pmod{691}, \tag{57}$$

Open question. Lehmer (1943) asked whether $\tau(n) \neq 0$ for all $n \geq 1$; it is true for $n \leq 10^{15}$

4.4 Extra Notes

Explicit recap on dimension of cusp forms of weight 12. There are two spaces:

$$\mathcal{M}_{12} = \text{modular forms of weight-12}, \qquad \mathcal{S}_{12} = \text{cusp forms of weight 12}.$$

Since $\mathcal{M} \cong \mathbb{C}[E_4, E_6]$, the weight-12 piece has basis $\{E_4^3,\ E_6^2\}$, so $\dim \mathcal{M}_{12} = 2$. A cusp form must have zero constant term. Because both E_4^3 and E_6^2 start with constant term 1, the only way to cancel the constant term is their difference so:

$$\mathcal{S}_{12} = \mathbb{C} \cdot (E_4^3 - E_6^2) = \mathbb{C} \cdot \Delta,$$

and $\dim \mathcal{S}_{12} = 1$.

Values of $H(z)$.

$$H_2(z) = \sum_{n,m}{}' \frac{1}{(m-1+nz)(m+nz)}, \qquad H(z) = \sum_{m,n}{}' \frac{1}{(m-1+nz)(m+nz)},$$

with the prime indicating the omissions and the fixed order (sum in m first, then n). Use

$$\frac{1}{(m-1+nz)(m+nz)} = \frac{1}{m-1+nz} - \frac{1}{m+nz}.$$

(i) $n = 0$ slice.

$$\sum_{m \in \mathbb{Z} \setminus \{0,1\}} \left(\frac{1}{m-1} - \frac{1}{m} \right) = 2,$$

hence both H and H_2 receive a contribution $= 2$ from $n = 0$.

(ii) $n \neq 0$ slices. For fixed $n \neq 0$,

$$\sum_{m \in \mathbb{Z}} \left(\frac{1}{m-1+nz} - \frac{1}{m+nz} \right) = \pi \big(\cot \pi(1-nz) - \cot \pi(nz) \big) = 0,$$

using $\sum_{m \in \mathbb{Z}} \frac{1}{m+a} = \pi \cot(\pi a)$ and $\cot(\pi(1-w)) = -\cot(\pi w)$. Thus all $n \neq 0$ slices cancel *formally*.

(iii) Conditional summation correction. With the prescribed order of summation and omissions, the $n \neq 0$ part of H acquires the standard correction $-\frac{2\pi i}{z}$ (from comparing $\sum_m \frac{1}{m+a}$ at a and $a-1$ near an integer). No such correction appears in H_2 due to the extra $1/m$ factor.

$$\boxed{H_2(z) = 2, \qquad H(z) = 2 - \frac{2\pi i}{z}}.$$

Logarithmic differential of F. Let $q = e^{2\pi i z}$ and $F(z) = q\prod_{n\geq 1}(1-q^n)^{24}$. Taking logs and differentiating,

$$\log F = \log q + 24\sum_{n\geq 1}\log(1-q^n),$$

$$\frac{dF}{F} = d(\log F) = \frac{dq}{q} + 24\sum_{n\geq 1}\frac{-\,nq^{n-1}\,dq}{1-q^n} = \frac{dq}{q}\Big(1 - 24\sum_{n\geq 1}\frac{nq^n}{1-q^n}\Big).$$

Using $\frac{q^n}{1-q^n} = \sum_{m\geq 1} q^{nm}$ and regrouping terms by $r = nm$,

$$\sum_{n\geq 1}\frac{nq^n}{1-q^n} = \sum_{n,m\geq 1} n\,q^{nm} = \sum_{r\geq 1}\Big(\sum_{n\,|\,r} n\Big)q^r = \sum_{r\geq 1}\sigma_1(r)\,q^r.$$

Hence

$$\boxed{\frac{dF}{F} = \frac{dq}{q}\Big(1 - 24\sum_{n=1}^{\infty}\sigma_1(n)\,q^n\Big).} \tag{47}$$

From the known expansion

$$G_1(z) = \frac{\pi^2}{3} - 8\pi^2\sum_{n=1}^{\infty}\sigma_1(n)\,q^n,$$

and $dq/q = 2\pi i\,dz$, we get

$$\frac{dF}{F} = (2\pi i\,dz)\Big(1 - 24\sum_{n\geq 1}\sigma_1(n)\,q^n\Big) = \frac{6i}{\pi}\Big(\frac{\pi^2}{3} - 8\pi^2\sum_{n\geq 1}\sigma_1(n)\,q^n\Big)\,dz$$

$$= \boxed{\frac{6i}{\pi}\,G_1(z)\,dz.} \tag{48}$$