Modular Forms Undergraduate Seminar Fall 2025

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If you spot any errors, please let me know.

1.1 Organizational Stuff

- Email: austin.lei@columbia.edu. Feel free to email me if you have any questions or concerns. I will try to respond to you within one or two business days.
- Website for the course. Updates will be posted here, so please check back regularly!
- Tentatively, our course schedule looks like the following:
  - Friday, September 12th: Meeting #2.
  - Starting week of September 15th: Meet at Monday 4:30-5:30 pm, and Wednesday 4-5 pm.

  If you have scheduling conflicts with this time, please let me know.
- Grading scheme:
  - Students will be expected to give a roughly equal number of talks. Assuming 7-8 students in the seminar, giving 1 talk will give a C, giving 2 talks will give a B, and giving 3 talks will give an A.
  - After 2 unexcused absences, each further unexcused absence will lower one's grade by a grade boundary (i.e. A to A-, or A- to B+). Please email me in advance if you have a conflict for a meeting.
  - While not required, providing notes for your own lecture or taking notes for the class for someone else's lecture will bump one's grade by a grade boundary (i.e. A- to A, or A to A+).
- A spreadsheet of talks/notes signups is here. Please avoid signing up for more than 3 talks until everyone has an opportunity to sign up for 3 talks. Topics are not listed for later talks – a list of potential ideas is listed below, but you are welcome to talk about anything provided it is related to modular forms – please talk with me to confirm!

1.2 What is a modular form?

As I put in the blurb for the course, modular forms are holomorphic functions that satisfy a lot of "symmetries". What does this actually mean? To start, let's define the simplest example of a modular form.

Definition 1.1. *We denote the **upper-half plane** by* $\mathbb{H}$*:*

$$\mathbb{H} := \{x + iy \in \mathbb{C} : y > 0\}.$$

We can define a group action on $\mathbb{H}$.

Definition 1.2. *The group* $\mathrm{SL}_2(\mathbb{R})$ *is the group of real matrices with determinant 1. Similarly,* $\mathrm{SL}_2(\mathbb{Z})$ *is the group of real matrices with determinant 1. Moreover,*

$$\mathrm{PSL}_2(\mathbb{R}) := \mathrm{SL}_2(\mathbb{R}) / \{\pm I_2\}.$$

Finally, $\mathrm{PSL}_2(\mathbb{Z}) := \mathrm{SL}_2(\mathbb{Z}) / \{\pm I_2\}$ *is the **modular group**. By abuse of notation, we often are sloppy between referring to* SL_2 *and* PSL_2*.*

We can define an action $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$: for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$, and $z \in \mathbb{H}$, we have the action

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \circ z = \frac{az+b}{cz+d}.$$

Here are some facts about this action.

- The action is well-defined; i.e. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \circ z \in \mathbb{H}$.

  Proof. One can show that $\mathrm{Im}(\alpha z) = \frac{\mathrm{Im}(z)}{|cz+d|^2}$. □

- The action is continuous with respect to the standard topologies.
- The action is **transitive**; there is an $\alpha \in \mathrm{SL}_2(\mathbb{R})$ sending any z to any z'.

  Proof. One can send i to z via $\begin{pmatrix} \sqrt{y} & \frac{x}{\sqrt{y}} \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix}$. □

- Defining

  $$\mathrm{PSL}_2(\mathbb{R}) := \mathrm{SL}_2(\mathbb{R}) / \{\pm I_2\},$$

  the action of $\mathrm{PSL}_2(\mathbb{R})$ on $\mathbb{H}$ is **faithful**; i.e. for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{R})$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \circ z = z$ for all $z \in \mathbb{H}$ implies $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = I_2$. Equivalently, for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \circ z = z$ for all $z \in \mathbb{H}$ implies $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm I_2$.

  Proof. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \circ z = z$ is a quadratic in z, so if equal for all z then $a = d = \pm 1$ and $b = c = 0$. □

While we can define modular forms for more general groups, for now we will restrict our focus to the simplest group $\mathrm{SL}_2(\mathbb{Z})$. It is in fact a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$; i.e. the neighborhoods around any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ embedded into $\mathrm{SL}_2(\mathbb{R})$ are finite.

Remark 1.3. *Why do we care about discrete subgroups? It will turn out when* Γ *is a discrete subgroup of* $\mathrm{SL}_2(\mathbb{Z})$*, then the quotient space* $\Gamma \backslash \mathbb{H}$ *will have a nice (Hausdorff) topology. Modular forms will arise from studying these quotients, so having a nice topology will be good.*

Definition 1.4. *A* ***modular form*** *for* $\mathrm{SL}_2(\mathbb{Z})$ *is a function* $f : \mathbb{H} \to \mathbb{C}$ *such that for any* $z \in \mathbb{H}$.

- f *is holomorphic on* $\mathbb{H}$.
- f *is holomorphic at the cusp at infinity; i.e.* f *is bounded as* $z \to i\infty$.
- *For some positive integer* k, f *satisfies the transformation property*

  $$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

  for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.

Here k *is the* ***weight*** *of the modular form.*

I'm being quite vague here in the definitions; I'll define what cusps mean later. For now, think of the first 2 conditions together as one unified holomorphicity condition. If you are not familiar with complex analysis, you should think of being holomorphic as being complex infinitely differentiable function – we'll give a crash course of the important complex analysis results you'll need later.

Remark 1.5. *Is there motivation for the $(cz+d)^k$ factor? If one assumes that*

$$f(\gamma z) = j(\gamma, z) f(z)$$

for some function j of $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and $z \in \mathbb{H}$, then one can get $(cz+d)^k$ as a factor naturally. This will come up in the discussion of automorphy factors.

1.3 Why should we care about modular forms?

Here is a list of potential interesting applications/connections that could be of interest for future talks.

- Diophantine equations - for example, one can count the number of solutions to $n = x^2 + y^2 + z^2 + w^2$ via modular forms.
- Elliptic curves - the famous modularity conjecture (now a theorem!) states that all elliptic curves correspond to a specific type of modular form – this was used to prove Fermat's Last Theorem.
- Combinatorics - Modular forms will have a q-expansion (i.e. Fourier expansion), and the coefficients can be interpreted to have combinatoric results. The classic example are the Ramunjan congruences; for example, Ramunjan found that

  $$p(5n+4) \equiv 0 \pmod{5},$$

  which can be proved using modular forms.
- Sphere packing - See this survey article by Cohn for more details. The work of Viazovska and others for sphere-packing in dimensions 8 and 24. The choice of function to optimize a sphere-packing bound turns out (almost magically) to be a modular form.
- String theory - See here.
- Some other potential places to look: here and here.

Some other topics could be excellent choices for talks:

- Poincare series, towards the Petersson trace formula (see Milne or Iwaniec, as listed in the references for the course)
- Algebraic geometric view of modular forms - modular forms arose out of the study of the geometry of certain modular curves. Milne would be a good resource for this.
- Representation theory - modular forms can be interpreted in the language of representations of adelic groups. This also leads to the generalization of modular forms for higher rank groups $\mathrm{SL}_n(\mathbb{Z})$ (Maass forms).
- Maass forms, but classically - For $\mathrm{SL}_2(\mathbb{Z})$ Maass forms, in particular, Chapter 3 of Goldfeld is an approachable place to start.
- L-functions of modular forms and the converse theorem - Bump Sections 1.3/1.5 is one place to look, although a bit terse.
- Rankin-Selberg Method - Bump Section 1.6 is one place to look, although a bit terse.

If any of these topics interest you in particular, please let me know and I can share more references!