Topics in Analytic Number Theory Notes

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These notes were taken in the Spring 2025 version of the Topics in Analytic Number Theory Class, taught by Dorian Goldfeld. If you spot any mistakes, please let me know.

Contents

1 Lecture 1 - 1/21/25

The real content will start on January 30th. There are colloquium talks on Thursday and next Tuesday - you are strongly recommmended to attend.
Some of the content in the course will follow his book *Automorphic Forms and L-Functions for the Group $GL(n,\mathbb{R})$*. When I refer to "Dorian's book" in the notes, this is the book I refer to.

1.1 History of Analytic Number Theory

- 1700s - Euler invents the zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$. Discovers the Euler product
$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}.$$
Gets the functional equation for ζ in special cases (like for $s = i$).
- 1859 - Riemann gets the functional equation for all s; letting
$$\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s),$$
with
$$\Gamma(s) = \int_0^{\infty} e^{-u}u^s \frac{\mathrm{d}u}{u},$$
he proves the functional equation
$$\xi(s) = \xi(1 - s).$$
How? Riemann uses the known identity
$$\sum_{n=-\infty}^{\infty} e^{-\pi n^2 y} = \frac{1}{\sqrt{y}} \sum_{n=-\infty}^{\infty} e^{-\pi n^2/y}$$
then applies the Mellin transform: for a smooth function $f : \mathbb{R}_{\geq 0} \to \mathbb{C}$, the Mellin transform is $\widetilde{f}(s) = \int_0^{\infty} f(y)y^s \frac{\mathrm{d}y}{y}$. (This arises from a change of variable from the Fourier transform). More specifically, you have to take
$$\int_0^{\infty} \left(\sum_{n=-\infty}^{\infty} e^{-\pi n^2 y} - 1 \right) y^s \frac{\mathrm{d}y}{y} = \pi^{-s}\Gamma(s)\zeta(2s).$$
- Dirichlet, 1800s: Taking $\chi : (\mathbb{Z}/q\mathbb{Z})^* \to \mathbb{C}^*$, you get the Dirichlet L function $L(s,\chi) = \sum_{n=1}^{\infty} \chi(n)/n^s$. Shows that everything one can do with the zeta function can be applied to L-function. Can use them to show that there are infinitely many primes in an arithmetic progression.
- Hecke, early 1900s: Generalizes previous exponential sums to theta functions
$$\theta(z) = \sum_{n=-\infty}^{\infty} e^{2\pi i n^2 z},$$
where $z = x + iy \in \mathbb{H}$. This function turns out to be modular: for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \Gamma_0(4)$,
$$\theta\left(\frac{az + b}{cz + d}\right) = \varepsilon_d^{-1}\chi_c(d)\sqrt{cz + d}\,\theta(z),$$
where χ_c is a Dirichlet character mod c and $\varepsilon_d = 1$ if $d \equiv 1 \pmod 4$ and -1 if $d \equiv -1 \pmod 4$.

Hecke looks at modular functions: Recall that for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$, a modular function f satisfies

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z),$$

for $k \in \mathbb{Z}_{>0}$. This implies $f(z+1) = f(z)$, giving periodicity in the x direction. If f is holomorphic, we get a Fourier expansion

$$f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}.$$

Hecke defines the Hecke L-function

$$L(s,f) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

We also have $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$, taking $z \mapsto -1/z$. This corresponds to taking y to $1/y$. This gives a functional equation for Hecke L-functions, with a symmetry on the completed L-functions taking $s \to k - s$.

Moreover, using Hecke operators, Hecke was able to show that Hecke L-functions have a Euler product. Everything Hecke does can be generalized to subgroups of $\mathrm{SL}_2(\mathbb{Z})$.

- Gelfand, Piatetski-Shapiro: Replace the upper half plane with matrices: points $x + iy$ are replaced with $\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}$, where $x \in \mathbb{R}$ and $y > 0$, and examine functions of the matrices: $f(z)$ is replaced by $f\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right)$ and $f\left(\frac{az+b}{cz+d}\right) = f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right)$. How can you do this? Will be explained later. They also introduced automorphic representations.

  This course will primarily focus on $\mathrm{SL}(n,\mathbb{Z})$, especially when $n \geq 3$. Hence the matrix approach becomes necessary.

- Jacquet-Godement: Introduced analogue of Hecke L-functions for cuspidal automorphic forms for higher rank. Lots of results due to Shalika-Jacquet-Piatetski-Shapiro.

- Eisenstein series: Selberg proves analytic continuation and functional equation (proof involves Fredholm operators). Langlands generalizes Selberg's proof to arbitrary reductive groups. We will talk about Eisenstein series for $\mathrm{SL}(n,\mathbb{Z})$ in this course.

1.2 Iwasawa decomposition for $\mathbf{GL}(n,\mathbb{R})$

Before we mentioned functions of matrices as a replacement for functions on $\mathbb{H}$. How do we make this work? Recall that a matrix $m \in M_n(\mathbb{R})$ is orthogonal if $m \cdot m^T = I$, or equivalently if all the rows/columns of m form an orthonormal basis. We denote the set of such matrices $O(n,\mathbb{R})$.

In particular, note that $O(2,\mathbb{R}) = \left\{ \begin{pmatrix} \pm\cos t & \mp\sin t \\ \pm\sin t & \pm\cos t \end{pmatrix} \right\}$.

Theorem 1.1 (Iwasawa). *Every $g \in GL(n,\mathbb{R})$ is of the form*

$$g = xykd,$$

where

- *x is an upper triangular matrix with 1s on the diagonal, whose elements are denoted x_{ij}, all real.*
- *y is a diagonal matrix, with $y_1 y_2 \ldots y_{n-1}$ in the top left, $y_1 y_2 \ldots y_{n-2}$ in the next entry, going down to 1 in the bottom right, with all the $y_i > 0$.*
- *$k \in O(n,\mathbb{R}) = K$, where K is used to denoted the maximal compact group.*

- *d is a diagonal matrix with d_0 on all entries on the diagonal, with $d_0 \neq 0$.*

Example 1.2. *In the $GL(2,\mathbb{R})$ case, the Iwasawa decomposition $g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} kd$. Hence we can express*

$$\mathbb{H} = GL(2,\mathbb{R})/(O(2,\mathbb{R}) \cdot \mathbb{R}^*).$$

In general, we get the generalized upper half plane

$$\mathfrak{h}^n := GL(n,\mathbb{R})/(O(n,\mathbb{R}) \cdot \mathbb{R}^*).$$

Proof. Recall that a positive definite matrix is a matrix $m \in M(n,\mathbb{R})$ such that m is symmetric and $xmx^T > 0$ for all nonzero $x \in \mathbb{R}^n$, or equivalently m is symmetric and all its eigenvalues are positive. Moreover, note that for any $u \in \mathrm{GL}(n,\mathbb{R})$, uu^T is positive definite.
Consider any $g \in \mathrm{GL}(n,\mathbb{R})$.

Claim 1.3. *There exists upper triangular matrix u, lower triangular matrix ℓ and diagonal matrix d, such that $ugg^T = \ell d$.*

Proof. View this as solving for u. There are $n(n-1)/2$ parameters for u and $n(n-1)/2$ equations (the upper elements of ℓd need to be 0). This can be solved because gg^T is full rank. ☐

This gives that

$$gg^T = u^{-1}\ell d = d\ell^T(u^T)^{-1},$$

hence $\ell du^T = ud\ell^T$. Note that the LHS is an lower triangular matrix, and the right is an upper triangular matrix, so $ud\ell^T = d^*$, some diagonal matrix.
Further manipulation gives that $ugg^T = d^*(u^T)^{-1}$, so $ugg^Tu^T = d^*$. The LHS must be positive definite, d^* must consist of positive entries on the diagonal. Let a be its squareroot. Then we can write

$$(aug)(aug)^T = I.$$

Hence $aug \in O(n,\mathbb{R})$, and hence we get the decomposition. ☐

Here is an alternative proof using Gram-Schmidt:

Proof. Let $a_1, \ldots, a_n$ be the column vectors of $g^{-1} \in GL(n,\mathbb{R})$ and $q_1, \ldots, q_n$ be the outputs of the Gram-Schmidt process for the a_i. Let q be the matrix with the q_i as columns. The Gram-Schmidt process gives us an upper triangular matrix r such that

$$g^{-1} = qr.$$

Taking the inverse precisely gives the Iwasawa decomposition for g, as desired. ☐

2 Lecture 2 - 1/30/25

Last time, we talked about the Iwasawa decomposition. We defined

$$\mathfrak{h}^n = \mathrm{GL}(n,\mathbb{R})/(O(n,\mathbb{R})\cdot\mathbb{R}^*),$$

and showed that every $g \in \mathfrak{h}^n$ has the decomposition

$$g = xy = \begin{pmatrix} 1 & & & & \\ & 1 & & x_{ij} & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \begin{pmatrix} y_1y_2\cdots y_{n-1} & & & & \\ & y_1y_2\cdots y_{n-2} & & & \\ & & \ddots & & \\ & & & y_1 & \\ & & & & 1 \end{pmatrix},$$

where the $x_{ij} \in \mathbb{R}$ and $y_j > 0$.

Example 2.1. *In the case $n = 2$, we have*

$$\mathfrak{h}^2 = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}.$$

This is isomorphic to the upper half place, with $z = x + iy$, $x \in \mathbb{R}$, $y > 0$.
This has complex structure, making it easier to study (holomorphic modular forms). However, $\mathfrak{h}^n$, for $n \geq 3$, has no complex structure.

2.1 GL$(n,\mathbb{Z})$ action on $\mathfrak{h}^n$

We have an action of $\mathrm{GL}(n,\mathbb{Z})$ acting on $\mathfrak{h}^n$, given via left-multiplication of matrices (modulo $O(n,\mathbb{R})\cdot\mathbb{R}^*$). This will be notated $\alpha \cdot g$, but sometimes I might be lazy and write it like pure multiplication.

Example 2.2. *Consider $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $g = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}$, or equivalently $z = x+iy$. (We will use g to denote elements of $\mathfrak{h}^n$, rather than z in Dorian's book. We will reserve z for the classical $n = 2$ upper half plane approach.) Then $\alpha z = \frac{az+b}{cz+d}$, and similarly for $\alpha \cdot g$*

$$\alpha \cdot g = \begin{pmatrix} ay & a+bx \\ cy & c+dx \end{pmatrix},$$

which we then need to quotient by the right element of $O(n,\mathbb{R})\cdot\mathbb{R}^*$ to get back into $\mathfrak{h}^n$.

The theory of automorphic forms is all about functions

$$f : \mathrm{GL}(n,\mathbb{Z})\backslash\mathfrak{h}^n \to \mathbb{C}.$$

Equivalently, for $\alpha \in \mathrm{GL}(n,\mathbb{Z})$, $g \in \mathfrak{h}^n$, $k \in K = O(n,\mathbb{R})$, and $d = \begin{pmatrix} d_0 & & & \\ & d_0 & & \\ & & \ddots & \\ & & & d_0 \end{pmatrix}$, for $d_0 = 0$, we want functions

$$f(\alpha g k d) = f(g).$$

Example 2.3. *When $n = 2$, this is precisely the theory of modular forms.*
In this case, we have the standard fundamental domain for $SL(2,\mathbb{Z})\backslash\mathfrak{h}^2$

$$\left\{z \in \mathfrak{h}^2 : |x| \leq 1/2, |z| \geq 1\right\}.$$

What is the area of this region? It is precisely the integral

$$\int_{-1/2}^{1/2}\int_{\sqrt{1-x^2}}^{\infty} \frac{\mathrm{d}x\,\mathrm{d}y}{y^2} = \frac{\pi}{3}.$$

Here $\frac{\mathrm{d}x\mathrm{d}y}{y^2}$ *is the hyperbolic measure. It is an* ***invariant measure****: it is invariant under the action* $z \mapsto \frac{az+b}{cz+d}$*. How does one show this? Note that we can write*

$$\frac{\mathrm{d}}{\mathrm{d}z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right),$$

so $\frac{\mathrm{d}}{\mathrm{d}z} = 1$ *and* $\frac{\mathrm{d}}{\mathrm{d}\overline{z}} = 0$*. Hence a holomorphic function can be defined as a function* $f : \mathbb{C} \to \mathbb{C}$ *such that* $\frac{\partial}{\partial \overline{z}} f = 0$*.*
Then we can express

$$\frac{\mathrm{d}x\,\mathrm{d}y}{y^2} = \frac{-i}{4}\frac{\mathrm{d}z \wedge \mathrm{d}\overline{z}}{\mathrm{Im}(z)^2},$$

where $\mathrm{d}z = \mathrm{d}x + i\,\mathrm{d}y$ *and* $\mathrm{d}\overline{z} = \mathrm{d}x - i\,\mathrm{d}y$*.*
Now, applying the action of $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\mathbb{R})$ *on the RHS and applying the quotient rule gives*

$$\frac{-i}{4}\frac{\mathrm{d}\frac{\alpha z+\beta}{\gamma z+\delta} \wedge \mathrm{d}\frac{\alpha\overline{z}+\beta}{\gamma\overline{z}+\delta}}{\mathrm{Im}\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right)^2} = -\frac{i}{4}\frac{\frac{\mathrm{d}z}{(\gamma z+\delta)^2} \wedge \frac{\mathrm{d}\overline{z}}{(\gamma\overline{z}+\delta)^2}}{\frac{\mathrm{Im}(z)^2}{|\gamma z+\delta|^4}} = \frac{-i}{4}\frac{\mathrm{d}z \wedge \mathrm{d}\overline{z}}{\mathrm{Im}(z)^2},$$

hence the measure is invariant.

We'll want to generalize this idea to $\mathrm{GL}(n)$, but this approach doesn't generalize naturally, since we lack complex structure.

2.2 Invariant measure on $\mathfrak{h}^n$

We will want to integrate $\mathrm{GL}(n,\mathbb{Z})$ invariant functions over $\mathfrak{h}^n$, so we need to define an invariant measure. Let $g = xy \in \mathfrak{h}^n$.

Proposition 2.4. *The measure*

$$\mathrm{d}g = \left(\prod_{1\le i<j\le n} \mathrm{d}x_{ij}\right)\left(\prod_{k=1}^{n-1} y_k^{-k(n-k)-1}\,\mathrm{d}y_k\right)$$

is invariant under $g \mapsto \alpha g$ *with* $\alpha \in GL(n,\mathbb{R})$*.*

Proof. It suffices to prove that measure is invariant for a set of generators for $\mathrm{GL}(n,\mathbb{R})$. In particular, $\mathrm{GL}(n,\mathbb{R})$ is generated by matrices B_n, W_n, D_n, where B_n are upper triangular matrices, W_n is the Weyl group of $\mathrm{GL}(n,\mathbb{R})$ (the set of all matrices in $\mathrm{GL}(n,\mathbb{Z})$ with precisely one 1 in each column and row), and

$$D_n = \begin{pmatrix} a_1a_2\dots a_{n-1} & & & & \\ & a_1a_2\dots a_{n-2} & & & \\ & & \ddots & & \\ & & & a_1 & \\ & & & & 1 \end{pmatrix}$$

are diagonal matrices.

Remark 2.5. *Why this notation for the diagonal matrices? Since we quotient out by* $\mathbb{R}^*$*, we can have the lower right element be 1. The formulas are all nicer with the* a_i *written this way. (There's also intuition involving root systems that Dorian doesn't want to get into.)*

First, we check the invariance under the action by D_n. Let $\alpha = \begin{pmatrix} a_1a_2\ldots a_{n-1} & & & & \\ & a_1a_2\ldots a_{n-2} & & & \\ & & \ddots & & \\ & & & a_1 & \\ & & & & 1 \end{pmatrix}$.

For any $g = xy$, we can write $\alpha g = (\alpha x \alpha^{-1})(\alpha y)$, where $(\alpha x \alpha^{-1})$ is an upper triangular matrix with 1s on the diagonal and

$$(\alpha x \alpha^{-1})_{ij} = \left(\prod_{k=n-j+1}^{n-i} a_k \right) x_{ij}$$

for all $i < j$, and

$$(\alpha y)_{ii} = \prod_{k=1}^{n-i} (\alpha_k y_k).$$

Plugging everything in, the a_k will all cancel, giving the desired invariance.
Dorian leaves the invariance by the upper triangular matrices and Weyl elements to the reader. Alternatively, details can be found in his book (Section 1.5). ☐

2.3 Siegel's theorem for the volume of the fundamental domain

Let $\Gamma_n = \mathrm{SL}(n, \mathbb{Z})$.

Theorem 2.6 (Siegel, 1936)**.**

$$Vol(\Gamma_n \backslash \mathfrak{h}^n) = n 2^{n-1} \prod_{\ell=2}^{n} \frac{\zeta(\ell)}{Vol(S^{\ell-1})},$$

where

$$Vol(S^{\ell-1}) = \frac{2(\sqrt{\pi})^\ell}{\Gamma\left(\frac{\ell}{2}\right)}.$$

The proof will require (a generalization of) the Poisson summation formula. Recall the standard Poisson summation formula:

Proposition 2.7 (Poisson summation)**.** *Let $f : \mathbb{R} \to \mathbb{C}$ be a smooth function (with some technical conditions, i.e. exponential decay). Then*

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \widehat{f}(n),$$

where $\widehat{f}(y) = \int_{-\infty}^{\infty} f(u) e^{-2\pi i y u} \, \mathrm{d}u$ is the Fourier transform.

Proof. Define the new function $G(x) = \sum_{n \in \mathbb{Z}} f(x+n)$. Note that $G(x+1) = G(x)$, so we have a Fourier expansion

$$G(x) = \sum_{k \in \mathbb{Z}} A_k e^{2\pi i k x}$$

where

$$A_k = \int_0^1 G(u) e^{-2\pi i u k} \, \mathrm{d}u \, .$$

Hence

$$\begin{aligned} G(x) &= \sum_{k \in \mathbb{Z}} \left(\int_0^1 \sum_{n \in \mathbb{Z}} f(u+n) e^{-2\pi i u k} \, \mathrm{d}u \right) e^{2\pi i k x} \\ &= \sum_{k \in \mathbb{Z}} \int_{-\infty}^{\infty} f(u) e^{-2\pi i u (k-x)} \, \mathrm{d}u \, , \end{aligned}$$

so we conclude that

$$\sum_{n\in\mathbb{Z}} f(x+n) = \sum_k \widehat{f}(k-x).$$

Substituting $x = 0$ gives the result. □

In particular, we will need a GL(2) version of Poisson summation.

Proposition 2.8 (Poisson summation for $\mathrm{GL}(2,\mathbb{R})$)**.** *Consider a smooth, compactly supported function $f : \mathbb{R}^2/SO(2,\mathbb{R}) \to \mathbb{C}$; i.e. $f((u,v)k) = f((u,v))$ for any $(u,v) \in \mathbb{R}^2$ and $k \in K = SO(2,\mathbb{R})$. Then we have*

$$\sum_{(m,n)\in\mathbb{Z}} f((m,n)\cdot g) = \sum_{(m,n)\in\mathbb{Z}^2} \widehat{f}((m,n)\cdot(g^T)^{-1}).$$

Here $\widehat{f}$ is the (double) Fourier transform

$$\widehat{f}((x,y)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f((u,v))e^{-2\pi ixu}e^{-2\pi iyv}\,\mathrm{d}u\,\mathrm{d}v\,.$$

Proof. Consider $g \in \mathrm{SL}(2,\mathbb{R})$ of the form $g = \begin{pmatrix} y^{1/2} & xy^{-1/2} \\ 0 & y^{-1/2}\end{pmatrix}$. We define

$$F(g) := \sum_{(m,n)\in\mathbb{Z}^2} f((m,n)\cdot g) = \sum_{(m,n)\in\mathbb{Z}^2} f(my^{1/2}, mxy^{-1/2} + ny^{-1/2}),$$

and for fixed g and n, define

$$G_g(n) := \sum_{m\in\mathbb{Z}} f(my^{1/2}, mxy^{-1/2} + ny^{-1/2}).$$

By standard Poisson Summation (in n),

$$F(g) = \sum_{n\in\mathbb{Z}} G_g(n) = \sum_{n\in\mathbb{Z}} \widehat{G_g}(n).$$

Hence

$$F(g) = \sum_{(m,n)\in\mathbb{Z}} \widehat{f}(my^{1/2}, mxy^{-1/2} + ny^{-1/2}) = \sum_{(m,n)\in\mathbb{Z}^2} \int_{-\infty}^{\infty} f(my^{1/2}, mxy^{-1/2} + uy^{-1/2})e^{-2\pi iun}\,\mathrm{d}u\,,$$

where above the Fourier transform is taken only in the n variable.
We now do the same thing in the m variable. Define

$$H_g(m) := \sum_{n\in\mathbb{Z}} \int_{-\infty}^{\infty} f(my^{1/2}, mxy^{-1/2} + uy^{-1/2})e^{-2\pi iun}\,\mathrm{d}u\,.$$

Poisson summation again gives that

$$F(g) = \sum_{m\in\mathbb{Z}} H_g(m) = \sum_{m\in\mathbb{Z}} \widehat{H_g}(m).$$

Hence, we can write

$$F(g) = \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \int_{-\infty}^{\infty} f(vy^{1/2}, vxy^{-1/2} + uy^{-1/2})e^{-2\pi inu}e^{-2\pi imv}\,\mathrm{d}u\,\mathrm{d}v\,.$$

Making the transformation $u' = vy^{1/2}$ and $v' = vxy^{-1/2} + uy^{-1/2}$ finishes the proof. □

We'll get to Siegel's proof next time.

Remark 2.9. *Siegel's proof for the volume of the fundamental domain was generalized by Langlands in the paper The volume of the fundamental domain for some arithmetic subgroups of Chevalley groups, Proc AMS, 1965.*

3 Lecture 3 - 2/4/25

3.1 Fundamental Domains

Consider a topological space X and group G, with G acting on X. Recall that a (left) group action is a map $\circ : G \times X \to X$ such that $e \circ x = x$ for all x, and $(g_1 g_2) \circ x = g_1 \circ (g_2 \circ x)$.

Proposition 3.1. *$GL(n, \mathbb{Z})$ acts on $\mathfrak{h}^n = GL(n, \mathbb{R})/(O(n, \mathbb{R}) \cdot \mathbb{R}^*)$. If $\gamma \in GL(n, \mathbb{Z})$ and $g \in \mathfrak{h}^n$, $\gamma \circ g := \gamma \cdot g$ as matrix multiplication.*

Proof. This is clear. □

Note that

$$\mathfrak{h}^n = \mathrm{GL}(n, \mathbb{R})/(O(n, \mathbb{R}) \cdot \mathbb{R}^*) = \mathrm{SL}(n, \mathbb{R})/SO(n, \mathbb{R}).$$

Hence we can talk about the action of $\mathrm{SL}(n, \mathbb{Z})$ on $\mathfrak{h}^n = \mathrm{SL}(n, \mathbb{R})/SO(n, \mathbb{R})$ (via matrix multiplication). What is a fundamental domain for this action?
Recall that a fundamental domain for G acting on X, typically denoted $G \backslash X$), has the properties

- Every $x \in X$ is equivalent to some $y \in G \backslash X$, where $x = g \circ y$ for some $g \in G$.
- No two points in the fundamental domain are equivalent to each other.

In the $n = 2$ case, we have the standard fundamental domain

$$\mathrm{SL}(2, \mathbb{Z}) \backslash \mathfrak{h}^2 = \left\{ z = x + iy \in \mathfrak{h}^2 \mid |x| \leq \frac{1}{2}, |z| \geq 1 \right\}.$$

To generalize this idea, we will consider a Siegel set:

$$\Sigma_{\frac{\sqrt{3}}{2}, \frac{1}{2}} = \left\{ x + iy \in \mathfrak{h}^2 \mid |x| \leq \frac{1}{2}, y \geq \frac{\sqrt{3}}{2} \right\}.$$

This set is bigger than the fundamental domain, but small enough to be a good approximation for analytic purposes. Specifically,

$$\bigcup_{\gamma \in \mathrm{SL}(2, \mathbb{Z})} \gamma \cdot \Sigma_{\frac{\sqrt{3}}{2}, \frac{1}{2}} = \mathfrak{h}^2.$$

Theorem 3.2 (Siegel)**.** *The Siegel set for $SL(n, \mathbb{Z}) \backslash \mathfrak{h}^n$*

$$\Sigma_{\frac{\sqrt{3}}{2}, \frac{1}{2}} = \left\{ xy \in \mathfrak{h}^n \mid |x_{ij}| \leq \frac{1}{2}, y \geq \frac{\sqrt{3}}{2} \right\}$$

satisfies

$$\bigcup_{\gamma \in SL(n, \mathbb{Z})} \gamma \cdot \Sigma_{\frac{\sqrt{3}}{2}, \frac{1}{2}} = \mathfrak{h}^n.$$

The proof can be found in Dorian's book.

3.2 Volume of fundamental domain $\mathrm{SL}(2, \mathbb{Z}) \backslash \mathfrak{h}^2$

Last time we stated

Theorem 3.3 (Siegel, 1936)**.**

$$Vol(\Gamma_n \backslash \mathfrak{h}^n) = n 2^{n-1} \prod_{\ell=2}^{n} \frac{\zeta(\ell)}{Vol(S^{\ell-1})},$$

where

$$Vol(S^{\ell-1}) = \frac{2(\sqrt{\pi})^\ell}{\Gamma\left(\frac{\ell}{2}\right)}.$$

The proof is inductive, so we'll want to prove the statement for $n = 2$.

Proof for $n = 2$. Let $K = O(2, \mathbb{R})$. Consider a smooth and compactly supported function $f : \mathbb{R}^2/K \to \mathbb{C}$. We can then define

$$F(g) = \sum_{(m,n)\in\mathbb{Z}^2} f((m,n)\cdot g),$$

where multiplication is taken as a row vector multiplied by a matrix. Since f is right-invariant by K, we have that

$$F(gk) = F(g)$$

for all $g \in \mathrm{GL}(2, \mathbb{R})$ and $k \in K$.

Claim 3.4. *$F(\gamma g) = F(g)$ for all $\gamma \in SL(2, \mathbb{Z})$.*

Proof. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$. Since we want $g \in \mathrm{SL}(2, \mathbb{R})$, we take $g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix}$. Then

$$\begin{aligned}
F(\gamma g) &= \sum_{(m,n)} f\left((m,n)\begin{pmatrix} a & b \\ c & d \end{pmatrix} g\right) \\
&= \sum_{(m,n)} F\left((am+cn, bm+dn)\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix}\right) \\
&= \sum_{M,N} F\left((M,N)\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix}\right) = F(g)
\end{aligned}$$

which proves the claim. Here there are no convergence issues because f has compact support. □

Next, letting $\Gamma = \mathrm{SL}(2, \mathbb{Z})$, consider

$$\int_{\Gamma\backslash\mathfrak{h}^2} F(g)\,\mathrm{d}g = \int_{\Gamma\backslash\mathfrak{h}^2} F\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix}\right)\frac{\mathrm{d}x\,\mathrm{d}y}{y^2}.$$

Again, this integral converges because f is compactly supported.
Note that we can write

$$\{(m,n) \mid m,n \in \mathbb{Z}\} = \{(0,0)\} \cup \bigcup_{\substack{\ell=1 \\ \gamma\in\Gamma_\infty\backslash\Gamma}}^{\infty} \{\ell(0,1)\gamma\},$$

where $\Gamma_\infty = \left\{\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} \mid r \in \mathbb{Z}\right\}$. This follows because

$$\Gamma_\infty\backslash\Gamma = \left\{\begin{pmatrix} * & * \\ c & d \end{pmatrix} \mid (c,d) = 1\right\}.$$

Thus,

$$\begin{aligned}
\int_{\Gamma\backslash\mathfrak{h}^2} F(g)\,\mathrm{d}g &= \int_{\Gamma\backslash\mathfrak{h}^2} F(0,0)\,\mathrm{d}g + \int_{\Gamma\backslash\mathfrak{h}^2} \sum_{\ell=1}^{\infty} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} f(\ell(0,1)\gamma g)\,\mathrm{d}g \\
&= F((0,0))\cdot\mathrm{Vol}(\Gamma\backslash\mathfrak{h}^2) + 2\int_{\Gamma_\infty\backslash\mathfrak{h}^2} \sum_{\ell=1}^{\infty} f((0,\ell)\cdot g)\,\mathrm{d}g\,,
\end{aligned}$$

where the factor of 2 arises because $-I_2 \in \Gamma$ fixes $\mathfrak{h}^2$.

Hence

$$\begin{aligned}\int_{\Gamma\backslash\mathfrak{h}^2} F(g)\,\mathrm{d}g &= F((0,0))\cdot \mathrm{Vol}(\Gamma\backslash\mathfrak{h}^2) + 2\int_{\Gamma_\infty\backslash\mathfrak{h}^2}\sum_{\ell=1}^{\infty} f\left((0,\ell)\cdot\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}y^{1/2} & 0\\ 0 & y^{-1/2}\end{pmatrix}\right)\frac{\mathrm{d}x\,\mathrm{d}y}{y^2}\\ &= F((0,0))\cdot \mathrm{Vol}(\Gamma\backslash\mathfrak{h}^2) + 2\int_{\Gamma_\infty\backslash\mathfrak{h}^2}\sum_{\ell=1}^{\infty} f\left((0,\ell y^{-1/2})\right)\frac{\mathrm{d}x\,\mathrm{d}y}{y^2}\\ &= F((0,0))\cdot \mathrm{Vol}(\Gamma\backslash\mathfrak{h}^2) + 2\int_{x=0}^{1}\int_{y=0}^{\infty}\sum_{\ell=1}^{\infty} f\left((0,\ell y^{-1/2})\right)\frac{\mathrm{d}x\,\mathrm{d}y}{y^2}.\end{aligned}$$

Taking the transformations $y \mapsto \ell^2 y$ in the first line and then $y \to y^{-2}$ in the second line, we get

$$\begin{aligned}2\int_{x=0}^{1}\int_{y=0}^{\infty}\sum_{\ell=1}^{\infty} f\left((0,\ell y^{-1/2})\right)\frac{\mathrm{d}x\,\mathrm{d}y}{y^2} &= 2\int_{x=0}^{1}\int_{y=0}^{\infty}\sum_{\ell=1}^{\infty} f\left((0,y^{-1/2})\right)\frac{1}{\ell^2}\frac{\mathrm{d}x\,\mathrm{d}y}{y^2}\\ &= 4\zeta(2)\int_0^\infty f\left((0,y)\right) y\,\mathrm{d}y\,.\end{aligned}$$

Now, we convert to polar coordinates. Since f is right invariant by $k = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix} \in K$,

$$f((0,y)) = f((y\sin\theta, y\cos\theta))$$

for any θ.
Thus we get that

$$\begin{aligned}4\zeta(2)\int_{y=0}^{\infty} f\left((0,y)\right) y\,\mathrm{d}y &= \frac{2\zeta(2)}{\pi}\int_0^{2\pi}\int_{y=0}^{\infty} f\left((y\sin\theta, y\cos\theta)\right) y\,\mathrm{d}y\,\mathrm{d}\theta\\ &= \frac{2\zeta(2)}{\pi}\int_{\mathbb{R}^2} f(u,v)\,\mathrm{d}u\,\mathrm{d}v = \frac{2\zeta(2)}{\pi}\widehat{f}((0,0)).\end{aligned}$$

Hence we have shown that

$$\int_{\Gamma\backslash\mathfrak{h}^2} F(g)\,\mathrm{d}g = f((0,0))\mathrm{Vol}(\Gamma\backslash\mathfrak{h}^2) + \frac{2\zeta(2)}{\pi}\widehat{f}((0,0)).$$

Now, consider replacing f by $\widehat{f}$. By Poisson summation for $\mathrm{GL}(2,\mathbb{R})$,

$$\sum_{(m,n)\in\mathbb{Z}^2} f((m,n)g) = \sum_{(m,n)\in\mathbb{Z}^2}\widehat{f}((m,n)(g^T)^{-1}).$$

We can replace g by $(g^T)^{-1}$ in all of the computation above, and nothing would change. Hence, we get that

$$\int_{\Gamma\backslash\mathfrak{h}^2} F(g)\,\mathrm{d}g = \widehat{f}((0,0))\mathrm{Vol}(\Gamma\backslash\mathfrak{h}^2) + \frac{2\zeta(2)}{\pi} f((0,0)),$$

using that $\widehat{\widehat{f}}(x) = f(-x)$. Subtracting the two equations and solving for the volume gives the desired formula. ∎

Next time, we will finish the proof for general n.

4 Lecture 4 - 2/6/25

4.1 Volume of fundamental domain $\mathbf{SL}(n,\mathbb{Z})\backslash\mathfrak{h}^n$

This time we will finish the proof of Siegel's theorem:

Theorem 4.1 (Siegel, 1936)**.**

$$Vol(\Gamma_n\backslash\mathfrak{h}^n) = n2^{n-1}\prod_{\ell=2}^{n}\frac{\zeta(\ell)}{Vol(S^{\ell-1})},$$

where

$$Vol(S^{\ell-1}) = \frac{2(\sqrt{\pi})^\ell}{\Gamma\left(\frac{\ell}{2}\right)}.$$

We proved it for $n = 2$ last time. Now we will finish the proof for $n > 2$ inductively.
We will use the Poisson summation formula for $\mathrm{GL}(n,\mathbb{R})$:

Proposition 4.2. *For a function $f : \mathbb{R}^n/K_n \to \mathbb{C}$, where $K_n = O(n,\mathbb{R})$, we have that*

$$\sum_{m\in\mathbb{Z}^n} f(m\cdot g) = \sum_{m\in\mathbb{Z}^n}\widehat{f}(m\cdot(g^T)^{-1}).$$

We showed this for $n = 2$; it can be generalized to higher n inductively.

Proof of Siegel's Theorem. For more details, one can check Dorian's book, section 1.6.
Let $\Gamma_n = \mathrm{SL}(n,\mathbb{Z})$. Recall that for $g \in \mathfrak{h}^n$, we write $g = xy$ with the usual notation for x and y. We want to this to lie in $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n,\mathbb{R})$, so we instead consider

$$y = \begin{pmatrix} y_1y_2\cdots y_{n-1}t & & & & \\ & y_1y_2\cdots y_{n-2}t & & & \\ & & \ddots & & \\ & & & y_1t & \\ & & & & t\end{pmatrix},$$

where $t = \left(\prod_{j=1}^{n-1} y_j^{n-j}\right)^{-1}$.
Emulating the proof for $n = 2$, let $f : \mathbb{R}^n/K_n \to \mathbb{C}$ be a smooth and compactly supported function. Again we define

$$F(g) = \sum_{m\in\mathbb{Z}^n} f(m\cdot g).$$

Then we can show $F(\gamma g) = F(g)$ for all $\gamma \in \mathrm{SL}(n,\mathbb{Z})$.

Definition 4.3. *The* ***mirabolic subgroup of GL***(n) *is*

$$P_n = \left\{\begin{pmatrix} & & * & & \\ 0 & 0 & \dots & 0 & 1\end{pmatrix}\right\}$$

Then one can check that

$$F(g) = f((0,\dots,0)) + \sum_{\ell=1}^{\infty}\sum_{\gamma\in P_n\backslash\Gamma_n} f(\ell\cdot e_n\cdot\gamma g),$$

where $e_n = (0,\dots,0,1)$.
Now, we have that

$$\begin{aligned}\int_{\Gamma_n\backslash\mathfrak{h}^n} F(g)\,\mathrm{d}g &= f((0,\dots,0))\mathrm{Vol}(\Gamma_n\backslash\mathfrak{h}^n) + \int_{\Gamma_n\backslash\mathfrak{h}^n}\sum_{\ell=1}^{\infty}\sum_{\gamma\in P_n\backslash\Gamma_n} f(\ell\cdot e_n\cdot\gamma g)\,\mathrm{d}g \\ &= f((0,\dots,0))\mathrm{Vol}(\Gamma_n\backslash\mathfrak{h}^n) + 2\sum_{\ell=1}^{\infty}\int_{P_n\backslash\mathfrak{h}^n} f(\ell\cdot e_n\cdot g)\,\mathrm{d}g\,,\end{aligned}$$

where the factor of 2 appears because $-I_n \in \Gamma_n$ fixes $\mathfrak{h}^n$, and $-I_n \in \Gamma_n$ for n even and $\in \emptyset(n,\mathbb{R})$ for n odd. Now we can write

$$g = \begin{pmatrix} 1 & & & & \\ & 1 & & x_{ij} & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \begin{pmatrix} y_1y_2\cdots y_{n-1}t & & & & \\ & y_1y_2\cdots y_{n-2}t & & & \\ & & \ddots & & \\ & & & y_1t & \\ & & & & t \end{pmatrix} \begin{pmatrix} t^{\frac{1}{n-1}} I_{n-1} & \\ & t \end{pmatrix} \begin{pmatrix} t^{-\frac{1}{n-1}} I_{n-1} & \\ & t \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & & & x_{1n} \\ & 1 & & & x_{2n} \\ & & \ddots & & \vdots \\ & & & 1 & x_{n-1,n} \\ & & & & 1 \end{pmatrix} \begin{pmatrix} g' & \\ & 1 \end{pmatrix} \begin{pmatrix} t^{-\frac{1}{n-1}} I_{n-1} & \\ & t \end{pmatrix}$$

where g' is the $n-1$ by $n-1$ matrix

$$g' = \begin{pmatrix} 1 & x_{12} & x_{13} & \dots & x_{1,n-1} \\ & 1 & x_{23} & \dots & x_{2,n-1} \\ & & \ddots & \ddots & \vdots \\ & & & 1 & x_{n-2,n-1} \\ & & & & 1 \end{pmatrix} \begin{pmatrix} y_1y_2\cdots y_{n-1}t^{n/(n-1)} & & & \\ & y_1y_2\cdots y_{n-2}t^{n/(n-1)} & & \\ & & \ddots & \\ & & & y_1t^{n/(n-1)} \end{pmatrix} \in \mathfrak{h}^{n-1}$$

Recall that

$$\mathrm{d}g = \left(\prod_{1\le i<j\le n} \mathrm{d}x_{ij}\right) \prod_{k=1}^{n-1} y_k^{-k(n-k)-1}\,\mathrm{d}y_k\,,$$

and we have that

$$\mathrm{d}g' = \left(\prod_{1\le i<j\le n-1} \mathrm{d}x_{ij}\right) \prod_{k=1}^{n-2} y_{k+1}^{-k(n-k-1)-1}\,\mathrm{d}y_{k+1}\,.$$

Computation thus gives us that

$$\mathrm{d}g = -\frac{n}{n-1}\,\mathrm{d}g' \left(\prod_{j=1}^{n-1} \mathrm{d}x_{j,n}\right) t^n \frac{\mathrm{d}t}{t}.$$

Now, to apply induction, we will want to relate $P_n\backslash\mathfrak{h}^n$ to $\Gamma_{n-1}\backslash\mathfrak{h}^{n-1}$.
Every $p \in P_n$ is of the form

$$p = \begin{pmatrix} \gamma & b \\ & 1 \end{pmatrix} = \begin{pmatrix} I_{n-1} & b \\ & 1 \end{pmatrix} \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix}$$

with $\Gamma \in \mathrm{SL}(n-1,\mathbb{Z})$ and $b \in \mathbb{Z}^{n-1}$. Moreover, every $g \in \mathfrak{h}^n$ is of the form

$$g = \begin{pmatrix} g' & u \\ & 1 \end{pmatrix} \begin{pmatrix} t^{-\frac{1}{n-1}} I_{n-1} & \\ & t \end{pmatrix} = \begin{pmatrix} I_{n-1} & u \\ & 1 \end{pmatrix} \begin{pmatrix} g' & \\ & 1 \end{pmatrix} \begin{pmatrix} t^{-\frac{1}{n-1}} I_{n-1} & \\ & t \end{pmatrix},$$

where

$$u = \begin{pmatrix} u_{1,n} \\ u_{2,n} \\ \vdots \\ u_{n-1,n} \end{pmatrix}.$$

Then

$$p \cdot g = \begin{pmatrix} I_{n-1} & b \\ & 1 \end{pmatrix} \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} \begin{pmatrix} I_{n-1} & u \\ & 1 \end{pmatrix} \begin{pmatrix} g' & \\ & 1 \end{pmatrix} \begin{pmatrix} t^{-\frac{1}{n-1}} I_{n-1} & \\ & t \end{pmatrix}.$$

Let $U_n(\mathbb{Z})$ denote matrices with 1s on the diagonal, integers in the right most column, and 0s elsewhere, and similarly for $U_n(\mathbb{R})$.

Lemma 4.4. *Fix a $\gamma \in SL(n-1, \mathbb{Z})$. We have an action of $U_n(\mathbb{Z})$ on $\mathbb{R}^{n-1}$ given by left multiplication of $U_n(\mathbb{Z})$ on $\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} \cdot U_n(\mathbb{R})$, with fundamental domain given by*

$$\left\{ \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & & & & u_1 \\ & 1 & & & u_2 \\ & & \ddots & & \vdots \\ & & & 1 & u_{n-1} \\ & & & & 1 \end{pmatrix} \mid 0 \le u_i < 1 \right\}$$

Moreover,

$$U_n(\mathbb{Z}) \backslash \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} U_n(\mathbb{R}) \cong (\mathbb{Z} \backslash \mathbb{R})^{n-1}.$$

Proof. One can write

$$\begin{aligned} \bigcup_{m \in \mathbb{Z}^{n-1}} \begin{pmatrix} I_{n-1} & m \\ & 1 \end{pmatrix} \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} \begin{pmatrix} I_{n-1} & (\mathbb{Z}\backslash\mathbb{R})^{n-1} \\ & 1 \end{pmatrix} &= \bigcup_{m \in \mathbb{Z}^{n-1}} \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} \begin{pmatrix} I_{n-1} & \gamma^{-1} m \\ & 1 \end{pmatrix} \begin{pmatrix} I_{n-1} & (\mathbb{Z}\backslash\mathbb{R})^{n-1} \\ & 1 \end{pmatrix} \\ &= \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} \bigcup_{m \in \mathbb{Z}^{n-1}} \begin{pmatrix} I_{n-1} & (\mathbb{Z}\backslash\mathbb{R})^{n-1} + \gamma^{-1} m \\ & 1 \end{pmatrix} \\ &= \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} U_n(\mathbb{R}). \end{aligned}$$

□

Hence, examining our expression $p \cdot g$ and applying the lemma, we get the decomposition

$$P_n \backslash \mathfrak{h}^n \cong (\mathrm{SL}(n-1, \mathbb{Z}) \backslash \mathfrak{h}^{n-1}) \times (\mathbb{Z}\backslash\mathbb{R})^{n-1} \times (0, \infty).$$

Moreover, note that

$$f(\ell e_n g) = f\left(\ell e_n \begin{pmatrix} g' & u \\ & 1 \end{pmatrix} \begin{pmatrix} t^{-\frac{1}{n-1}} I_{n-1} & \\ & t \end{pmatrix} \right) = f(\ell t e_n).$$

Thus we can write

$$2 \sum_{\ell=1}^{\infty} \int_{P_n \backslash \mathfrak{h}^n} f(\ell \cdot e_n \cdot g) \, \mathrm{d}g = 2 \frac{n}{n-1} \sum_{\ell=1}^{\infty} \left(\int_{\Gamma_{n-1} \backslash \mathfrak{h}^{n-1}} \mathrm{d}g' \right) \left(\int_{(\mathbb{Z}\backslash\mathbb{R})^{n-1}} \prod_{i=1}^{n-1} \mathrm{d}x_{i,n} \right) \left(\int_0^{\infty} f(\ell t e_n) t^n \frac{\mathrm{d}t}{t} \right).$$

By induction, the first integral on the RHS is the volume $\Gamma_{n-1} \backslash \mathfrak{h}^{n-1}$. The second integral is 1. Thus, it suffices to compute the third integral.
Making a transformation $t \to \frac{t}{\ell}$, we have that

$$\sum_{\ell=1}^{\infty} \int_0^{\infty} f(\ell t e_n) t^n \frac{\mathrm{d}t}{t} = \zeta(n) \int_0^{\infty} f(\ell t e_n) t^n \frac{\mathrm{d}t}{t}.$$

Lemma 4.5.

$$\int_0^{\infty} f(\ell t e_n) t^n \frac{\mathrm{d}t}{t} = \frac{\widehat{f}((0, \ldots, 0))}{Vol(S^{n-1})}.$$

Proof. Use the n dimensional spherical coordinates

$$\begin{aligned} x_1 &= t(\sin\theta_{n-1})\cdots(\sin\theta_2)(\sin\theta_1) \\ x_2 &= t(\sin\theta_{n-1})\cdots(\sin\theta_2)(\cos\theta_1) \\ &\vdots \\ x_{n-1} &= t(\sin\theta_{n-1})(\cos\theta_{n-2}) \\ x_n &= t\cos\theta_{n-1} \end{aligned}$$

In particular, note that $x_1^2 + \cdots + x_n^2 = 1$. We have the invariant measure on S^{n-1}

$$\mathrm{d}\theta = \prod_{1\le j<n} (\sin\theta_j)^{j-1}\,\mathrm{d}j\,,$$

so

$$\mathrm{d}x_1\cdots\mathrm{d}x_n = t^{n-1}\,\mathrm{d}t\,\mathrm{d}\theta\,.$$

This measure is invariant under rotations, so

$$f((0,\ldots,0,t)) = \frac{1}{\mathrm{Vol}(S^{n-1})}\int_{S^{n-1}} f(x_1,\ldots,x_n))\,\mathrm{d}\theta\,,$$

and thus

$$\int_0^\infty f((0,\ldots,0,t))t^n\frac{\mathrm{d}t}{t} = \frac{1}{\mathrm{Vol}(S^{n-1})}\int_{\mathbb{R}^n} f(x_1,\ldots,x_n)\,\mathrm{d}x_1\ldots\mathrm{d}x_n = \widehat{f}((0,\ldots,0)),$$

where we apply polar coordinates. □

This gives the formula

$$\int_{\Gamma_n\backslash\mathfrak{h}^n} F(g)\,\mathrm{d}g = f((0,\ldots,0))\mathrm{Vol}(\Gamma_n\backslash\mathfrak{h}^n) + 2\frac{n}{n-1}\cdot\frac{\widehat{f}((0,\ldots,0))}{\mathrm{Vol}(S^{n-1})}\cdot\mathrm{Vol}(\Gamma_{n-1}\backslash\mathfrak{h}^{n-1}).$$

Now, we repeat the same process replacing $\widehat{f}$ and f, using the Poisson summation formula. Since the computation remains the same with $(g^T)^{-1}$ in place of g, we get that

$$\int_{\Gamma_n\backslash\mathfrak{h}^n} F(g)\,\mathrm{d}g = \widehat{f}((0,\ldots,0))\mathrm{Vol}(\Gamma_n\backslash\mathfrak{h}^n) + 2\frac{n}{n-1}\cdot\frac{f((0,\ldots,0))}{\mathrm{Vol}(S^{n-1})}\cdot\mathrm{Vol}(\Gamma_{n-1}\backslash\mathfrak{h}^{n-1}).$$

Choosing f such that $f((0,\ldots,0)) \neq \widehat{f}((0,\ldots,0))$ and manipulating the two equations gives the inductive formula

$$\mathrm{Vol}(\Gamma_n\backslash\mathfrak{h}^n) = 2\frac{n}{n-1}\cdot\frac{1}{\mathrm{Vol}(S^{n-1})}\cdot\mathrm{Vol}(\Gamma_{n-1}\backslash\mathfrak{h}^{n-1})$$

which finishes the theorem. □

Next time, we start the theory of automorphic forms.

5 Lecture 5 - 2/11/25

Today, we will briefly review the theory of GL(2) automorphic forms using Langlands parameters. A reference for more details is Dorian's Chapter 3, but be aware that everything needs to be converted from spectral to Langlands parameters.

5.1 Laplacian for GL(2)

Recall that $\mathfrak{h}^2 = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{R}, y > 0 \right\}$.

Definition 5.1. *We have the* ***(hyperbolic) Laplacian***

$$\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

For the real line, the Laplacian is $\frac{\mathrm{d}^2}{\mathrm{d}x^2}$, which is invariant for $x \mapsto x+1$. Analagously, the hyperbolic Laplacian behaves similarly.

Proposition 5.2. *Δ is invariant under $SL(2,\mathbb{R})$: for all $\gamma \in SL(2,\mathbb{R})$, $\Delta g = \Delta(\gamma g)$.*

We will discuss a proof later involving Lie theory, but we'll start with a classical proof, which doesn't generalize to higher rank, since for higher rank we no longer have the complex structure.

Proof. Recall that we have

$$\frac{\mathrm{d}}{\mathrm{d}z} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right)$$

such that $\frac{\partial}{\partial z} = 1$, and

$$\frac{\mathrm{d}}{\mathrm{d}\overline{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$$

that kills all holomorphic functions.
Note that we can write

$$\Delta = 4 \operatorname{Im}(z) \frac{\mathrm{d}}{\mathrm{d}z} \frac{\mathrm{d}}{\mathrm{d}\overline{z}}.$$

For any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mathrm{SL}(2,\mathbb{R})$, we have that

$$\frac{\mathrm{d}}{\mathrm{d}\frac{az+b}{cz+d}} = (cz+d)^2 \frac{\mathrm{d}}{\mathrm{d}z}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\overline{\frac{az+b}{cz+d}}} = (c\overline{z}+d)^2 \frac{\mathrm{d}}{\mathrm{d}\overline{z}},$$

and since

$$\operatorname{Im}\left(\frac{az+b}{cz+d} \right) = \frac{1}{|cz+d|^2},$$

Δ remains invariant. $\square$

Remark 5.3. *Δ being $SL(2,\mathbb{Z})$-invariant implies that $\Delta f(\gamma g) = \Delta f(g)$ for any $\gamma \in SL(2,\mathbb{Z})$. This is good; we will want our operator to send automorphic functions to automorphic functions.*

5.2 Maass forms for $\mathbf{SL}(2,\mathbb{Z})$

We are now equipped to discussed automorphic forms on GL(2).

Definition 5.4. *A* ***Maass form for*** $\boldsymbol{SL}(2,\mathbb{Z})$ *is a smooth function* $f : \mathfrak{h}^2 \to \mathbb{C}$ *such that*

- *Automorphic condition:* $f(\gamma g) = f(g)$ *for all* $\gamma \in SL(2,\mathbb{Z})$ *and* $g \in GL(2,\mathbb{R})$*.*
- $\Delta f = \lambda f$ *for some* $\lambda \in \mathbb{C}$*.*
- *Growth condition:* $|f(g)| \ll y^{-B}$ *for some* $B > 0$ *and* $y \to \infty$*.*

Remark 5.5. *The only holomorphic function of this type are the constant functions. However, there exist non-holomorphic examples - namely the Maass forms.*

Remark 5.6. *We in fact can assume that* $\lambda \in \mathbb{R}^+$*, rather than in* $\mathbb{C}$*. To see this, we have an positive definite inner product for Maass forms*

$$\langle f_1, f_2 \rangle = \int_{SL(2,\mathbb{Z})\backslash \mathfrak{h}^2} f_1(g)\overline{f_2(g)} \frac{\mathrm{d}x\,\mathrm{d}y}{y^2}.$$

By applying Green's theorem, one can show that

$$\langle \Delta f, f \rangle = \langle f, \Delta f \rangle ,$$

so for f *a Maass form, we conclude that the* λ *are real and positive.*

In the previous definition, we say that λ is the **spectral parameter** of the Maass form f. Its corresponding **Langlands parameter** is $\alpha = (\alpha_0, -\alpha_0)$, where $\frac{1}{4} - \alpha_0^2 = \lambda$.
In general, on GL(n), a Langlands parameter is a vector of the form

$$(\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$$

where $\alpha_1 + \cdots + \alpha_n = 0$. We will be able to associate a Langlands parameter to every Maass form for SL($n, \mathbb{Z}$).
We now give a broad survey of results in the GL(2) theory.

Remark 5.7. *It is conjectured that there is only one Maass form (up to constant multiple) for each eigenvalue* λ*; the best known upper bound is* $\sqrt{\lambda}$*.*

Remark 5.8. *It is difficult to construct* $SL(2,\mathbb{Z})$ *Maass forms - the proof of existence was first shown using the Selberg trace formula. Moreover, it also shows that the set of Maass forms is countable.*

Conjecture 5.9 (Selberg Eigenvalue Conjecture)**.** $\lambda \geq \frac{1}{4}$.

Remark 5.10. *This bound is tight - one can construct Maass forms of eigenvalue* $\frac{1}{4}$ *using the Gelbart-Jacquet lift from* $GL(2)$ *to* $GL(3)$*.*
The conjecture can be proven when the fundamental domain is a triangle, so it has been proved for $SL(2,\mathbb{Z})$*. However, it is not proved for congruence subgroups of* $SL(2,\mathbb{Z})$*, or for higher rank. Weaker lower bounds have been shown, but getting the optimal* $\frac{1}{4}$ *bound would improve error terms in applications.*

Remark 5.11. *Due to a result by Luo-Rudnick-Sarnak, the Ramanujan conjecture on the Fourier coefficients of Maass forms should be seen as roughly the same difficulty as the Selberg Eigenvalue conjecture from the adelic/representation point of view.*

5.3 Fourier expansion of GL(2) Maass forms

Take $g = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}(2, \mathbb{R})$, and let f be a Maass form with Langlands parameter $\alpha = (\alpha_0, -\alpha_0)$. Note that

$$f(g) = f\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} g\right) = f\left(\begin{pmatrix} 1 & x+1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right),$$

so we have periodicity as $x \mapsto x+1$. This gives a Fourier expansion

$$f(g) = \sum_{n \neq 0} \int_0^1 f\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} g\right) e^{-2\pi i n u}\, \mathrm{d}u\,.$$

Note that there is no constant term because of the polynomial decay of the Maass form. We can define

$$W_n(g) = \int_0^1 f\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} g\right) e^{-2\pi i n u}\, \mathrm{d}u\,.$$

We note that this satsifies some nice properties:

- $\Delta W_n(g) = \left(\frac{1}{4} - \alpha_0^2\right) W_n(g)$
- $W_n\left(\begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} g\right) = e^{2\pi i n v} W_n(g)$.

Moreover, W_n inherits the growth properties of the Maass form.

Definition 5.12. *We say that any (smooth) function satsifying these three conditions is a* ***Whittaker function*** *with Langlands parameters α.*

In GL(2), we can write out these Whittaker functions explictly in terms of Bessel functions.
First, note that by the second property above, we can rewrite our Fourier expansion as

$$\begin{aligned} f(g) &= \sum_{n \neq 0} \int_0^1 f\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} g\right) e^{-2\pi i n u}\, \mathrm{d}u \\ &= \sum_{n \neq 0} \int_0^1 f\left(\begin{pmatrix} 1 & u+x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) e^{-2\pi i n u}\, \mathrm{d}u \\ &= \sum_{n \neq 0} \left(\int_0^1 f\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) e^{-2\pi i n u}\, \mathrm{d}u\right) e^{2\pi i n x}. \end{aligned}$$

Let

$$A_n(y) := \int_0^1 f\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) e^{-2\pi i n u}\, \mathrm{d}u\,.$$

Then our Fourier expansion can be written as

$$f(g) = \sum_{n \neq 0} A_n(y) e^{2\pi i n x},$$

where $A_n(y) e^{2\pi i n x} = W_n(g)$.
Let's examine the differential condition. We know that

$$\begin{aligned} \Delta f(g) &= \sum_{n \neq 0} \Delta A_n(y) e^{2\pi i n x} \\ &= \sum_{n \neq 0} -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(A_n(y) e^{2\pi i n x}\right) \\ &= \sum_{n \neq 0} -y^2 \left(A_n''(y) - 4\pi^2 n^2 A_n(y)\right) e^{2\pi i n x}. \end{aligned}$$

We also have that

$$\Delta f(g) = \left(\frac{1}{4} - \alpha^2\right) f(g).$$

Hence for each n we get the differential equation

$$\left(\frac{1}{4} - \alpha^2\right) A_n(y) = -y^2 \left(A_n''(y) - 4\pi^2 n^2 A_n(y)\right)$$

Because the Maass form (and hence the Whittaker function) has polynomial decay, the second order differential equation has a unique solution (up to constant) in terms of a Bessel function, giving the Fourier expansion

$$f(g) = \sum_{n \neq 0} a(n) \sqrt{2\pi y} K_\alpha(2\pi |n| y) e^{2\pi i n x},$$

where

$$K_\alpha(y) = \frac{1}{2} \int_0^\infty e^{-\frac{1}{2} y(u + 1/u)} u^\alpha \frac{\mathrm{d}u}{u}$$

is the Bessel function. We remark that this function has exponential decay in y.
To the Fourier expansion we associate the L-function

$$L_f(s) = \sum_{n=1}^\infty \frac{a(n)}{n^s}.$$

This L-function will have a functional equation and Euler product.

5.4 Eisenstein series for GL(2)

We briefly discuss the theory of Eisenstein series for GL(2).
We have the classical construction of the Eisenstein series

$$E(z,s) = \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}(2,\mathbb{Z})} (\operatorname{Im} \gamma z)^s$$

where $\Gamma_\infty = \left\{ \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \mid m \in \mathbb{Z} \right\}$.
We call $\operatorname{Im}(z)^s = y^s$ the **power function** - this will generalize for GL(n).
The Eisenstein series satsifies all the properties of a Maass forms except for the growth condition; instead, we have polynomial growth: $|E(x + iy, s)| \ll y^B(s)$ as $y \to \infty$ with $B(s) > 0$.
By normalizing by the proper gamma factors, and letting $E^*(z,s)$ be the normalized Eisenstein series, we get the functional equation

$$E^*(z,s) = E^*(z, 1-s).$$

Definition 5.13. *We say that an automorphic form (i.e. Eisenstein series or Maass form) f for $SL(2,\mathbb{Z})$ has **spectral parameters** $\lambda = s(1-s)$ if f has the same eigenvalues as y^s under Δ.*

We can also define the Eisenstein series in terms of Langlands parameters. Letting $\alpha = (\alpha_0, -\alpha_0)$, we have the corresponding power function $y^{\frac{1}{2}+\alpha_0}$ and thus corresponding Eisenstein series

$$E(\alpha, g) := \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}(2,\mathbb{Z})} (\operatorname{Im} \gamma z)^{\frac{1}{2}+\alpha_0}.$$

With the correct normalization by Gamma factors again, this gives the functional equation

$$E^*((\alpha_0, -\alpha_0), g) = E^*((-\alpha_0, \alpha_0), g),$$

in other words, we can replace α_0 with $-\alpha_0$ in the definition of the Eisenstein series and everything is the same.
To summarize, we reiterate the definition of a Maass form with Langlands parameters.

Definition 5.14. *A* ***Maass form with Langlands parameters*** $\alpha = (\alpha_0, -\alpha_0)$ *is a smooth function* $f : \mathfrak{h}^2 \to \mathbb{C}$ *such that*

- $f(\gamma g) = f(g)$ *for all* $\gamma \in SL(2,\mathbb{Z})$ *and* $g \in GL(2,\mathbb{R})$
- $\Delta f = \lambda f$*, where* λ *is the eigenvalue of the power function* $y^{\frac{1}{2}+\alpha_0}$ *under* Δ*.*
- $|f(g)| \ll y^{-B}$ *for some* $B > 0$ *and* $y \to \infty$*.*

This will be the definition of Maass form we use going forward.

5.5 Invariant differential operators on $\mathfrak{h}^n$

We want to generalize all of the above theory to GL(n), so we'll first need to describe the invariant differential operators on GL(n).
Recall the Iwasawa decomposition $g = xy$. An invariant differential operator will be some polynomial in $\frac{\partial}{\partial x_{ij}}$ and $\frac{\partial}{\partial y_k}$. To describe precisely what these are, we will turn to Lie theory.

Definition 5.15. *A* ***Lie algebra*** L *over a field* K *is a vector space over* K *equipped with the Lie bracket, a bilinear map* $[\cdot,\cdot] : L \times L \to L$ *such that for all* $a, b, c \in L$ *and* $\alpha, \beta \in K$*,*

- $[a, \alpha b + \gamma c] = \alpha[a,b] + \gamma[a,c]$
- $[a,a] = 0$
- $[a,b] = -[b,a]$.
- $[a,[b,c]] + [b,[c,a]] + [c,[a,b]] = 0$.

Example 5.16. *We have* $\mathfrak{gl}(n,\mathbb{R})$*, the Lie algebra of* $GL(n,\mathbb{R})$*, is the additive vector space of all* $n \times n$ *matrices in* $\mathbb{R}$*, with Lie bracket* $[a,b] = a \cdot b - b \cdot a$*.*

Now we can construct the differential operators. Let $\alpha \in \mathfrak{gl}(n,\mathbb{R})$. Define

$$D_\alpha f(g) := \frac{\partial}{\partial t} F(g \exp(t\alpha))|_{t=0},$$

where $\exp(t\alpha) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!}(t\alpha)^\ell$.
Next time, we will show that the ring generated by all the D_α is the universal enveloping algebra of $\mathfrak{gl}(n,\mathbb{R})$, and construct differential operators using this.

6 Lecture 6 - 2/13/25

6.1 Lie theory on $\mathfrak{gl}(n, \mathbb{R})$

We'll want to use the theory of Lie algebras to define invariant differential operators.

Definition 6.1. *An **associative algebra over a field** K is a vector space A over K such that*

- *we have an associative product $\cdot : A \times A \to A$ such that for all $a, b, c \in A$ and $\alpha, \beta, \gamma \in K$,*
$$a \cdot (\beta b + \gamma c) = \beta(a \cdot b) + \gamma(a \cdot c),$$
and
$$(\alpha a + \beta b) \cdot c = \alpha(a \cdot c) + \beta(b \cdot c).$$

Definition 6.2. *A **Lie algebra** is a vector space L over a field K with a Lie bracket $[\cdot, \cdot] : L \times L \to L$ such that for all $a, b, c \in L$ and $\beta, \gamma \in K$*

- $[a, \alpha b + \gamma c] = \alpha[a, b] + \gamma[a, c]$
- $[a, a] = 0$
- $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0.$

Given an associative algebra over K, we can construct a Lie algebra by defining $[a, b] = a \circ b - b \circ a$. One can check that this satisfies the definition of a Lie algebra.

Definition 6.3. *A **derivation on an associative algebra** A is a K-linear map $D : A \to A$ such that $D(xy) = (Dx) \cdot y + x \cdot (Dy)$ for all $x, y \in A$.*

Definition 6.4. *The derivations on an associative algebra form a Lie algebra, with Lie bracket*
$$[D, D'] = D \circ D' - D' \circ D,$$
where $\circ$ is composition.

Proof. To prove that this is a Lie algebra, we need to show
$$[D, D'](xy) = ([D, D']x) \cdot y + x \cdot ([D, D']y).$$

We have that
$$\begin{aligned}(D \circ D')(xy) &= D(D'(xy)) \\ &= D((D'x) \cdot y + x \cdot (D'y)) \\ &= D((D'x) \cdot y) + D(x \cdot (D'y)) \\ &= (D \circ D')x \cdot y + (D'x) \cdot (Dy) + (Dx) \cdot (D'y) + x \cdot (D' \circ D)y.\end{aligned}$$

Similarly,
$$(D' \circ D)(xy) = (D' \circ D)x \cdot y + (Dx) \cdot (D'y) + (D'x) \cdot (Dy) + x \cdot (D \circ D')y,$$
and subtract the two gives the desired result. □

Given an associative algebra A, we can construct an associated Lie algebra $L(A)$ (by the $a \circ b - b \circ a$ construction). It is also possible to start with $L(A)$ and go back and recover the associative algebra (the **universal enveloping algebra**). This can be constructed in general with universal properties, but we will only construct it for the specific case we care about.
We start with $\mathrm{GL}(n, \mathbb{R})$, a Lie group. Its corresponding Lie algebra is $\mathfrak{gl}(n, \mathbb{R})$, the vector space of $n \times n$ matrices over $\mathbb{R}$ with Lie bracket
$$[a, b] = a \cdot b - b \cdot a,$$

where the $\cdot$ denotes matrix multiplication. Motivated by the previous discussion, we want to construct a derivative on $\mathfrak{gl}(n, \mathbb{R})$.
Consider a smooth function $F : \mathrm{GL}(n, \mathbb{R}) \to \mathbb{C}$, and consider any $\alpha \in \mathrm{GL}(n, \mathbb{R})$. Then we define

$$D_\alpha F(g) := \frac{\partial}{\partial t} F(g \cdot e^{t\alpha})|_{t=0},$$

where t is a real variable and

$$e^{t\alpha} = \sum_{\ell=0}^{\infty} \frac{(t\alpha)^\ell}{\ell!}$$

is the matrix exponential. Since we only consider the first order partial derivative, this is equivalent to

$$D_\alpha F(g) = \frac{\partial}{\partial t} F(g + tg\alpha)|_{t=0}.$$

Example 6.5. *Let* $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathfrak{gl}(2, \mathbb{R})$*, let* $F(g) = F\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = a + 2b + c^2 - 3d$*, and let* $\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$*. Then*

$$\begin{aligned} D_\alpha F\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) &= \frac{\partial}{\partial t} F\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} + t\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)|_{t=0} \\ &= \frac{\partial}{\partial t} F\left(\begin{pmatrix} a & b + at \\ c & d + ct \end{pmatrix}\right)|_{t=0} \\ &= 2a - 3c. \end{aligned}$$

Proposition 6.6 (Properties of D_α)**.**

- $D_{\alpha+\beta} = D_\alpha + D_\beta$
- $D_\alpha \circ D_\beta - D_\beta \circ D_\alpha = D_{[\alpha,\beta]}$

Proof. Treating F as a function of n^2 variables ($n \times n$ variables), we can apply the standard multivariate chain rule

$$D_\alpha F(g) = \frac{\partial}{\partial t}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(g + tg\alpha)_{ij} \cdot \frac{\partial}{\partial g_{ij}} F(g + tg\alpha)\right)|_{t=0} = \sum_{i=1}^{n}\sum_{j=1}^{n}(g\alpha)_{ij} \cdot \frac{\partial}{\partial g_{ij}} F(g),$$

where $(g + tg\alpha)_{ij}$ denotes the i, jth element, and then we get the first equation.
Applying a similar computation for $D_\alpha \circ D_\beta - D_\beta \circ D_\alpha$ gives the desired result. □

With these properties, we have constructed the universal enveloping algebra of $\mathfrak{gl}(n, \mathbb{R})$, denoted $U(\mathfrak{gl}(n, \mathbb{R}))$, the algebra of all of these operators D_α. (We still need to show that the kernel of the map $\alpha \mapsto D_\alpha$ is trivial; see Goldfeld-Hundley Lemma 4.5.4.)

6.2 Center of the universal enveloping algebra of $\mathfrak{gl}(n, \mathbb{R})$

We want to find all elements $D \in U(\mathfrak{gl}(n, \mathbb{R}))$ that lie in the center $ZU(\mathfrak{gl}(n, \mathbb{R}))$; i.e. $[D, D'] = [D', D]$ for all $D' \in U(\mathfrak{gl}(n, \mathbb{R}))$. Why do we care?

Proposition 6.7. *Let* $F : SL(n, \mathbb{Z})\backslash\mathfrak{h}^n \to \mathbb{C}$ *an automorphic form; i.e.*

$$f(\gamma gkz) = f(g)$$

for all $\gamma \in SL(n, \mathbb{Z}), k \in K = O(n, \mathbb{R})$*,* $z \in Z_n$ *(diagonal elements with the same element along the diagonal). If* $D \in ZU(\mathfrak{gl}(n, \mathbb{R}))$*, then*

$$(Df)(\gamma gkz) = Df(g)$$

for all γ, k, z*; i.e.* D *will send automorphic forms to automorphic forms.*

Proof. For all $D \in U(\mathfrak{gl}(n, \mathbb{R}))$, we have that

$$(DF)(\gamma g z) = DF(g).$$

In particular,

$$(D_\alpha F)(\gamma g z) = \frac{\partial}{\partial t} F(\gamma g z e^{t\alpha})|_{t=0} = \frac{\partial}{\partial t} F(g e^{t\alpha} z)|_{t=0} = \frac{\partial}{\partial t} F(g e^{t\alpha})|_{t=0} = DF(g),$$

using that z commutes with everything in $\mathrm{GL}(n, \mathbb{R})$.
It remains to prove that $(DF)(gk) = DF(g)$ for $k \in K = O(n, \mathbb{R})$. Since $-I_n \in Z_n$, it suffices to prove the condition for $k \in SO(n, \mathbb{R})$.
Let $h \in \mathfrak{gl}(n, \mathbb{R})$, such that $h + h^T = 0$; i.e. $h \in \mathfrak{so}(n, \mathbb{R})$. For $u \in \mathbb{R}$, define

$$\phi(u) := D(f(ge^{uh})) - (Df)(ge^{uh}).$$

Note that $\phi(0) = 0$. We want to show that $\phi \equiv 0$; it is sufficient to show that $\phi'(u) = 0$ for all u.
We have that

$$\begin{aligned}
\phi'(u) &= \frac{\partial}{\partial t} \phi(u+t)|_{t=0} \\
&= \frac{\partial}{\partial t} \left(D(f(ge^{(u+t)h})) - (Df)(ge^{(u+t)h}) \right) |_{t=0} \\
&= \frac{\partial}{\partial t} \left(D(f(ge^{uh}e^{th})) - (Df)(ge^{uh}e^{th}) \right) |_{t=0} \\
&= (D(D_h f))(ge^{uh}) - (D_h(Df))(ge^{uh}) = 0,
\end{aligned}$$

using that $D \circ D_h = D_h \circ D$ since D is in the center.
Hence $\phi(u) = 0$ for all $u \in U$. Moreover, note that $e^{uh} \in \mathrm{SO}(n, \mathbb{R})$. Hence, we conclude that DF is invariant under $\mathrm{SO}(n, \mathbb{R})$, so we are done. ☐

Next time, we will construct Casimir elements, which lie in the center of the universal enveloping algebra. The construction will involve considering $D_{ij} := D_{E_{ij}}$, where E_{ij} is a 1 at position i, j and 0 elsewhere. This construction for GL(2) will recover the Laplacian.

7 Lecture 7 - 2/18/25

7.1 Casimir Operators on GL$(n, \mathbb{R})$

Last time, we talked about $\mathfrak{gl}(n, \mathbb{R})$, the Lie algebra of $\mathrm{GL}(n, \mathbb{R})$, which is the space of all $n \times n$ matrices over $\mathbb{R}$ with Lie bracket $[\alpha, \beta] = \alpha\beta - \beta\alpha$. Associated to $\mathfrak{gl}(n, \mathbb{R})$ is its universal enveloping algebra $U(\mathfrak{gl}(n, \mathbb{R}))$. Concretely, $U(\mathfrak{gl}(n, \mathbb{R}))$ is the algebra generated by differential operators

$$D_\alpha F(g) = \frac{\partial}{\partial t} F(ge^{t\alpha})\Big|_{t=0} = \frac{\partial}{\partial t} F(g + tg\alpha)\Big|_{t=0}$$

for $F : \mathrm{GL}(n, \mathbb{R}) \to \mathbb{C}$.
We are interested in the center of the universal enveloping algebra $ZU(\mathfrak{gl}(n, \mathbb{R}))$. In particular, we'd like to explicitly construct elements of the universal enveloping algebra.
For $1 \le i, j \le n$, let $E_{i,j}$ be the $n \times n$ matrix with 1 at i, j and 0 elsewhere.

Lemma 7.1. *$[E_{i,j}, E_{i',j'}] = \delta_{i',j} E_{i,j'} - \delta_{i,j'} E_{i',j}$, where $\delta_{i,j}$ is 1 if $i = j$ and 0 otherwise.*

The proof is simple and hence omitted.
For each i, j, define $D_{i,j} := D_{E_{i,j}}$.

Definition 7.2 (Casimir Differential Operator)**.** *For each $2 \le m \le n$, a* ***Casimir differential operator*** *is of the form*

$$D = \sum_{i_1=1}^n \sum_{i_2=1}^n \cdots \sum_{i_m=1}^n D_{i_1,i_2} D_{i_2,i_3} \dots D_{i_m,i_1}.$$

Theorem 7.3. *For any Casimir differential operator, $D \in ZU(\mathfrak{gl}(n, \mathbb{R}))$.*

Proof. We prove the theorem for $m = 2$. The theorem generalizes for higher m.
It suffices to show that for all $1 \le r, s \le n$,

$$D_{r,s} D = D D_{r,s},$$

or equivalently $[D_{r,s}, D] = 0$.

Lemma 7.4. *For any $D \in U(\mathfrak{gl}(n, \mathbb{R}))$ and $\alpha, \beta \in \mathfrak{gl}(n, \mathbb{R})$,*

$$[D_\alpha, D_\beta D] = [D_\alpha, D_\beta] D + D_\beta [D_\alpha, D].$$

Proof.

$$\begin{aligned}
[D_\alpha, D_\beta D] &= D_\alpha D_\beta D - D_\beta D D_\alpha \\
&= D_\alpha D_\beta D - D_\beta D_\alpha D - D_\beta D D_\alpha + D_\beta D_\alpha D \\
&= [D_\alpha, D_\beta] D + D_\beta [D_\alpha, D].
\end{aligned}$$

□

Now, we have D is a Casimir differential operator. Then by applying the Lemma,

$$\begin{aligned}
[D_{r,s}, D] &= \sum_{i_1=1}^n \sum_{i_2=1}^n [D_{r,s}, D_{i_1,i_2} D_{i_2,i_1}] \\
&= \sum_{i_1=1}^n \sum_{i_2=1}^n [D_{r,s}, D_{i_1,i_2}] D_{i_2,i_1} + D_{i_1,i_2} [D_{r,s}, D_{i_2,i_1}] \\
&= \sum_{i_1=1}^n \sum_{i_2=1}^n \left(\delta_{i_1,s} D_{r,i_2} - \delta_{r,i_2} D_{i_1,s}\right) D_{i_2,i_1} + D_{i_1,i_2} \left(\delta_{i_2,s} D_{r,i_1} - \delta_{r,i_1} D_{i_2,s}\right) \\
&= \sum_{i_2=1}^n D_{r,i_2} D_{i_2,s} - \sum_{i_1=1}^n D_{i_1,s} D_{r,i_1} + \sum_{i_1=1}^n D_{i_1,s} D_{r,i_1} - \sum_{i_2=1}^n D_{r,i_2} D_{i_2,s} \\
&= 0.
\end{aligned}$$

□

Now we try to compute the Casimir operator for GL(2).

Lemma 7.5. *Let* $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{R})$*. Then*

$$g = \begin{pmatrix} \frac{bc-ad}{c^2+d^2} & \frac{ac+bd}{c^2+d^2} \\ 0 & 1 \end{pmatrix} \pmod{O(2,\mathbb{R}) \cdot \mathbb{R}^*}.$$

In the GL(2) case, we have that

$$D = D_{1,1}D_{1,1} + D_{1,2}D_{2,1} + D_{2,1}D_{1,2} + D_{2,2}D_{2,2}.$$

We can compute that

$$D_{1,1}F(g) = \frac{\partial}{\partial t} F\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} + t\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right)\Bigg|_{t=0} = \frac{\partial}{\partial t} F\left(\begin{pmatrix} y(1+t) & x \\ 0 & 1 \end{pmatrix}\right)\Bigg|_{t=0} = y\frac{\partial}{\partial y} F\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right).$$

However, note that $D_{1,1}D_{1,1}$ is not simply composing $D_{1,1}$ twice, because $D_{1,1}$ may not be invariant under $O(n,\mathbb{R})$ (as it does not lie in the center). Thus we have to actually compute it explicitly.

$$\begin{aligned} D_{1,1}D_{1,1}F(g) &= \frac{\partial}{\partial t_1}\frac{\partial}{\partial t_2} F\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1+t_1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1+t_2 & 0 \\ 0 & 1 \end{pmatrix}\right)\Bigg|_{t_1=0,t_2=0} \\ &= \frac{\partial}{\partial t_1}\frac{\partial}{\partial t_2} F\left(\begin{pmatrix} y(1+t_1)(1+t_2) & x \\ 0 & 1 \end{pmatrix}\right)\Bigg|_{t_1=0,t_2=0} \\ &= \left(y\frac{\partial}{\partial y} + y^2\frac{\partial}{\partial y^2}\right) F(g). \end{aligned}$$

Similarly, we can compute that

$$\begin{aligned} D_{2,1}F(g) &= \frac{\partial}{\partial t} F\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} + t\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right)\Bigg|_{t=0} \\ &= \frac{\partial}{\partial t} F\left(\begin{pmatrix} y+tx & x \\ t & 1 \end{pmatrix}\right)\Bigg|_{t=0} \\ &= \frac{\partial}{\partial t} F\left(\begin{pmatrix} \frac{y}{t^2+1} & \frac{xt^2+yt+x}{t^2+1} \\ 0 & 1 \end{pmatrix}\right)\Bigg|_{t=0} \\ &= y\frac{\partial}{\partial x} F(g), \end{aligned}$$

where we transform via the Iwasawa decomposition, and

$$\begin{aligned} D_{2,1}D_{1,2}F(g) &= \frac{\partial}{\partial t_1}\frac{\partial}{\partial t_2} F\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ t_1 & 1 \end{pmatrix}\begin{pmatrix} 1 & t_2 \\ 0 & 1 \end{pmatrix}\right)\Bigg|_{t_1=0,t_2=0} \\ &= \left(-2y\frac{\partial}{\partial y} + y^2\frac{\partial^2}{\partial x^2}\right) F(g). \end{aligned}$$

Similar computations can be done to find that

$$D_{1,2}D_{2,1}F(g) = y\frac{\partial^2}{\partial x^2} F(g)$$

and

$$D_{2,2}D_{2,2}F(g) = \left(y\frac{\partial}{\partial y} + y^2\frac{\partial^2}{\partial y^2}\right) F(g).$$

Adding everything together gives $-\Delta$, which thus lies in the center, as desired.
Finally, it is enough to consider these Casimir operators:

Theorem 7.6 (Capelli, 1890)**.** *Let $n \geq 2$. Then $ZU(\mathfrak{gl}(n,\mathbb{R}))$ consists of all polynomials in the Casimir operators and D_{I_n}; it is a polynomial algebra of rank n in $\mathbb{R}$. Moreover, D_{I_n} annihilates all smooth functions $F : GL(n,\mathbb{R}) \to \mathbb{C}$ invariant under the center.*

In particular, note that if F is invariant under the center, then

$$D_{I_n}F(g) = \frac{\partial}{\partial t}F(g + tgI_n)\bigg|_{t=0} = \frac{\partial}{\partial t}F(g)\bigg|_{t=0} = 0.$$

For GL(3), there are 2 Casimir operators, which one can find written out explicitly in Dorian's book (Section 6.1).

7.2 Eigenfunctions of $ZU(\mathfrak{gl}(n,\mathbb{R}))$

Remember that for an automorphic function, we want it to be an eigenfunction of all of the $GL(n,\mathbb{R})$-invariant differential operators. Do these eigenfunctions even exist in general?
In the $n = 2$, we had that y^s was an eigenfunction. Analogously, we will construct a power function.

Definition 7.7 (Power function on $\mathfrak{h}^n$)**.** *The power function $I(g,\alpha)$, where $g = xy \in \mathfrak{h}^n$, is defined (formally) to be*

$$I(g,\alpha) := y^\alpha,$$

where $\alpha = (\alpha_1,\ldots,\alpha_n) \in \mathbb{C}^n$ is a Langlands parameter (i.e. $\alpha_1 + \cdots + \alpha_n = 0$.)

Supposing that $y = \begin{pmatrix} Y_1 & & & 0 \\ & Y_2 & & \\ & & \ddots & \\ 0 & & & Y_n \end{pmatrix}$, *we define*

$$y^\alpha := \prod_{i=1}^n Y_i^{\alpha_i}.$$

Note that for this definition to be well-defined, we need that $I(xy,\alpha) = I(y,\alpha)$ and $I(gkz,\alpha) = I(y,\alpha)$, for any upper triangular matrix g, $k \in K_n = O(n,\mathbb{R})$, and $z \in Z_n$ (the center). For invariance by the center to hold, we need that $\sum_i \alpha_i = 0$.

Definition 7.8 (Maass form for SL$(n,\mathbb{Z})$)**.** *A smooth function $F : \mathfrak{h}^n \to \mathbb{C}$ is a Maass form if*

- *$F(\gamma g) = F(g)$ for all $\gamma \in SL(n,\mathbb{Z})$ and $g \in \mathfrak{h}^n$.*
- *$|F(g)| \ll |y_1 y_2 \ldots y_{n-1}|^{-B}$ for some $B > 0$, where $g = xy$.*
- *$DF = \lambda_D F$ for all $D \in ZU(\mathfrak{gl}(n,\mathbb{R}))$. In particular, the λ_D should be the same as the eigenvalue for $I(g,\alpha+\rho)$, where $\rho = (\rho_1,\ldots,\rho_n)$, with $\rho_i = \frac{n+1}{2} - i$. In this case, we say that the Maass form has Langlands parameters α.*

The λ_D is called the Harish-Chandra character.

Remark 7.9. *ρ is half the sum of the positive roots in root system language.*

Example 7.10. *For $SL(2,\mathbb{Z})$, $\alpha = (\alpha_0, -\alpha_0) \in \mathbb{C}^2$ and $\rho = \left(\frac{1}{2}, -\frac{1}{2}\right)$. Then*

$$I(g,\alpha+\rho) = y^{\alpha_0 + \frac{1}{2}},$$

and

$$\Delta I(g,\alpha+\rho) = \left(\frac{1}{4} - \alpha_0^2\right) I(g,\alpha+\rho).$$

Selberg's eigenvalue conjecture is precisely that $\alpha_0 \in i\mathbb{R}$.

We will show next time the power function is actually an eigenfunction of all the differential operators, and then use the power functions to construct Eisenstein series (giving an example of an automorphic form).

8 Lecture 8 - 2/20/25

8.1 Power function is an eigenfunction of $ZU(\mathfrak{gl}(n,\mathbb{R}))$

Last time, we studied the $ZU(\mathfrak{gl}(n,\mathbb{R}))$, and constructed the Casimir elements. We then constructed the power function on $\mathfrak{h}^n$; for $g = \begin{pmatrix} Y_1 & & & * \\ & Y_2 & & \\ & & \ddots & \\ & & & Y_n \end{pmatrix}$ and Langlands parameter $\alpha = (\alpha_1, \dots, \alpha_n)$, we have that

$$I(g,\alpha) = y^\alpha := \prod_{i=1}^n Y_i^{\alpha_i}.$$

Proposition 8.1. *$DI(g,\alpha) = \lambda_D I(g,\alpha)$ for all $D \in ZU(\mathfrak{gl}(n,\mathbb{R}))$, where $\lambda_D \in \mathbb{C}$.*

Proof. We again only prove for $m = 2$; the proof generalizes.
If u is a unipotent matrix, then $I(ug,\alpha) = I(g,\alpha)$, as u only affects x within $g = xy$. Hence it suffices to consider the y component.
We have that

$$D_{i,i}I(g,\alpha) = \frac{\partial}{\partial t} I(y + tyE_{i,i}, \alpha)\bigg|_{t=0}.$$

In particular, note that

$$y + tyE_{i,i} = \begin{pmatrix} Y_1 & & & & & \\ & Y_2 & & & & \\ & & \ddots & & & \\ & & & Y_i(1+t) & & \\ & & & & \ddots & \\ & & & & & Y_n \end{pmatrix},$$

so

$$D_{i,i}I(g,\alpha) = \frac{\partial}{\partial t} Y_1^{\alpha_1} \dots (Y_i(1+t))^{\alpha_i} \dots Y_n^{\alpha_n}\bigg|_{t=0} = \alpha_i I(y,\alpha).$$

Moreover, for any ℓth power, one can show that

$$D_{i,i}I(g,\alpha) = \alpha_i^\ell I(g,\alpha).$$

Next, we consider $D_{i,j}I(g,\alpha)$. If $i < j$, then position (i,j) occurs above the diagonal, so

$$D_{i,j}I(g,\alpha) = \frac{\partial}{\partial t} I(y + tyE_{i,j}, \alpha)\bigg|_{t=0} = \frac{\partial}{\partial t} I(y,\alpha)\bigg|_{t=0} = 0,$$

as the t occurs above the diagonal, so it does not affect the y values (can be factored out with the unipotent part x).
It will suffice to only consider $i \le j$; see section 9.1 for more details.

□

Here we have a brief aside for Maass forms on $\mathrm{SL}(n,\mathbb{Z})$.
Let F be a Maass form for $\mathrm{SL}(n,\mathbb{Z})$. We have that $DF = \lambda_D F$, where λ_D is the same as the eigenvalue of $DI(g,\alpha+\rho) = \lambda_D I(g,\alpha+\rho)$.

Theorem 8.2 (Terras 1988, S. Miller)**.** *Let F be a Maass form for $SL(n,\mathbb{Z})$ with Langlands parameter $\alpha = (\alpha_1,\dots,\alpha_n)$, and let Δ be the Laplacian (the Casimir operator with $m = 2$). Then*

$$\lambda_\Delta = \frac{n^3 - n}{24} - \frac{1}{2}\sum_{i=1}^n \alpha_i^2.$$

Example 8.3. *When* $n = 2$*,* $\alpha = (\beta, -\beta)$*. Then*

$$\lambda_\Delta = \frac{1}{4} - \beta^2,$$

which matches what we knew previously.

We know that the Maass form are countable (by spectral theory on a Hilbert space), and we can order them by λ_Δ. It is conjectured that for $\mathrm{SL}(n, \mathbb{Z})$, given a specific choice of λ_Δ, there is only one Maass form (up to constant multiple) with eigenvalue λ_Δ.

8.2 Borel Eisenstein series

Let $B \subseteq GL(n, \mathbb{R})$ be the subset the set of upper triangular matrices in $\mathrm{GL}(n, \mathbb{R})$. Let $\Gamma = \mathrm{SL}(n, \mathbb{Z})$, and $\Gamma_B = \Gamma \cap B(\mathbb{Z})$ (upper triangular matrices with elements in $\mathbb{Z}$). Then we define the Borel Eisenstein series

$$E_B(g, \alpha) := \sum_{\gamma \in \Gamma_B \backslash \Gamma} I(\gamma g, \alpha + \rho),$$

where $\rho = (\rho_1, \ldots, \rho_n)$, with $\rho_i = \frac{n+1}{2} - i$. (The choice of ρ is to simplify the functional equation later.)

Proposition 8.4. *For* $Re(\alpha_i)$ *sufficiently large for all* $1 \leq i \leq n - 1$*,* $E_B(g, \delta)$ *converges absolutely and uniformly on compact subsets of* $\mathfrak{h}^n$*.*

Proof. Assume WLOG that all the α_i are real. We know that a fundamental domain for $\Gamma \backslash \mathfrak{h}^n$ is contained in

$$\Sigma_{\frac{\sqrt{3}}{2}, \frac{1}{2}} = \left\{ xy \mid |x| \leq \frac{1}{2}, y > \frac{\sqrt{3}}{2} \right\}.$$

It is enough to show that for any $g_0 \in \mathfrak{h}^n$ and small compact subset C_{g_0} containing g_0 that

$$\int_{C_{g_0}} |E(g, \alpha)| \, \mathrm{d}g \ll 1.$$

It is enough to prove that

$$\int_{C_{g_0}} \sum_{\gamma \in \Gamma_B \backslash \Gamma} |I(\gamma g, \alpha)| \, \mathrm{d}g \ll 1.$$

Note that there exist only finitely many $\gamma \in \Gamma_B \backslash \Gamma$ such that $\gamma g_0 \in \Sigma_{\frac{\sqrt{3}}{2}, \frac{1}{2}}$. Hence there exists a very large constant A such that $\gamma g_0 \notin \Sigma_{A, \frac{1}{2}}$. Hence, we can bound the RHS by

$$\int_0^1 \cdots \int_0^1 \int_0^A \cdots \int_0^A Y^\alpha \, \mathrm{d}g \,,$$

which is constant (uniformly in g_0), as desired. ∎

8.3 Parabolic Subgroups of $\mathrm{GL}(n, \mathbb{R})$

Consider any partition of $n = n_1 + \cdots + n_r$, with $1 \leq n_i \leq n$. We define the parabolic subgroup

$$\mathcal{P} = \mathcal{P}_{n_1, n_2, \ldots, n_r} := \left\{ \begin{pmatrix} \mathrm{GL}(n_1) & & & * \\ & \mathrm{GL}(n_2) & & \\ & & \ddots & \\ & & & \mathrm{GL}(n_r) \end{pmatrix} \right\}.$$

We also have the unipotent subgroup

$$N^{\mathcal{P}} := \left\{ \begin{pmatrix} I_{n_1} & & & * \\ & I_{n_2} & & \\ & & \ddots & \\ & & & I_{n_r} \end{pmatrix} \right\}.$$

The $\mathcal{P}$ factors as a product of the unipotent subgroup and the Levi subgroup

$$L^{\mathcal{P}} = \left\{ \begin{pmatrix} \mathrm{GL}(n_1) & & & 0 \\ & \mathrm{GL}(n_2) & & \\ & & \ddots & \\ & & & \mathrm{GL}(n_r) \end{pmatrix} \right\}.$$

In particular, $\mathcal{P} = N^{\mathcal{P}} L^{\mathcal{P}}$. Here we can think of the $N^{\mathcal{P}}$ as the x and $L^{\mathcal{P}}$ as the y in the typical Iwasawa decomposition $g = xy$.

Example 8.5. *Note that the Borel subgroup corresponds the partition $n = 1 + 1 + \cdots + 1$.*

Langlands was able to define Eisenstein series on parabolic subgroups. The idea will be to take in Maass forms on each of the $\mathrm{GL}(n_i)$ to induce the Eisenstein series.

8.4 Power Function on $\mathcal{P}_{n_1,\ldots,n_r}$

Consider any $g \in \mathcal{P}$, with

$$g = \begin{pmatrix} m_1 & & & * \\ & m_2 & & \\ & & \ddots & \\ & & & m_r \end{pmatrix},$$

where each $m_i \in \mathrm{GL}(n_i, \mathbb{R})$.

Definition 8.6. *Let $(s_1, s_2, \ldots, s_r) \in \mathbb{C}^r$, with $\sum_{i=1}^r n_i s_i = 0$. Then we define the power function $|\cdot|_{\mathcal{P}}^s : \mathcal{P} \to \mathbb{C}$ such that*

$$|g|_{\mathcal{P}}^s := \prod_{i=1}^r |\det(m_i)|^{s_i}.$$

Example 8.7. *For $n = 2$, $g = \begin{pmatrix} m_1 & * \\ & m_2 \end{pmatrix}$ for $m_1, m_2 \in \mathbb{R}$. Letting $s = (s_1, s_2)$ for $s_1 + s_2 = 0$, we have that*

$$|g|_{\mathcal{P}_{1,1}}^s = |m_1|^{s_1} \cdot |m_2|^{s_2}.$$

Proposition 8.8 (Properties of the power function)**.**

- *$|ug|_{\mathcal{P}}^s = |g|_{\mathcal{P}}^s$ for any $u \in N^p$.*
- *$|gk|_{\mathcal{P}}^s = |g|_{\mathcal{P}}^s$ for k with $k_1, \ldots, k_r$ along the diagonal, where $k_i \in O(n_i, \mathbb{R})$.*
- *Let $z \in Z(GL(n, \mathbb{R}))$. Then*

  $$|gz|_{\mathcal{P}}^s = |g|_{\mathcal{P}}^s.$$

  This is where the $\sum_{i=1}^r n_i s_i = 0$ is used.

Next time we will construct Eisenstein series on the parabolic subgroups using these power functions.

9 Lecture 9 - 2/25/25

9.1 Power function is an eigenfunction of $ZU(\mathfrak{gl}(n,\mathbb{R}))$

Last time, we talked about the proof that the $\mathrm{GL}(n,\mathbb{R})$ power function is an eigenfunction of all the $\mathrm{GL}(n,\mathbb{R})$ left-invariant differential operators, but the proof was incomplete. We complete the proof here.

Proof. Consider a Lie subgroup $G \subseteq \mathrm{GL}(n,\mathbb{R})$. Then we can construct a corresponding Lie algebra

$$\mathrm{Lie}(G) = \{\alpha \in M(n,\mathbb{R}) \mid e^{\alpha u} \in G \, \forall u \in \mathbb{R}\}.$$

There are two interesting subgroups coming from the Iwasawa decomposition. One of them is the Borel subgroup $B \subseteq \mathrm{GL}(n,\mathbb{R})$ of upper triangular matrices. The corresponding Lie algebra $\mathfrak{b}$ is all upper triangular $n \times n$ matrices with coefficients in $\mathbb{R}$. The other subgroup is $K = O(n,\mathbb{R})$. This has corresponding Lie algebra $\mathfrak{k}$ of skew-symmetric matrices; i.e. $n \times n$ matrices α such that $\alpha + \alpha^T = 0$.
On $\mathrm{GL}(n,\mathbb{R})$, we have the Iwasawa decomposition $\mathrm{GL}(n,\mathbb{R}) = B \cdot K$, so on the Lie algebras we have decomposition

$$M(n,\mathbb{R}) = \mathfrak{gl}(n,\mathbb{R}) = \mathfrak{b} \oplus \mathfrak{k}.$$

Proposition 9.1. *Let $D \in U(\mathfrak{gl}(n,\mathbb{R}))$. Then there exists $D^* \in U(\mathfrak{b})$ such that the action of D on a Maass form $F : \mathfrak{h}^n \to \mathbb{C}$ is the same as the action of D^* on F.*

Proof. Follows from the Iwasawa decomposition. Consider $\alpha \in M_n(\mathbb{R})$, decomposed into Iwasawa form $\alpha = \beta + \kappa$. Then

$$D_\alpha F(g) = \frac{\partial}{\partial t} F(ge^{t\alpha})\bigg|_{t=0} = \frac{\partial}{\partial t} F(ge^{t\beta})\bigg|_{t=0} = D_\beta F(g),$$

using that $e^{t\kappa} \in K$ and that F is right invariant by K. This applies for all D_α, and hence for $U(\mathfrak{gl}(n,\mathbb{R}))$. ☐

Thus, every $D \in ZU(\mathfrak{gl}(n,\mathbb{R}))$ can be expressed as the composition of D_β, with $\beta \in \mathfrak{b}$; i.e. the β are upper triangular matrices. Hence, we can write every D as a sum of compositions of $D_{i,j}$, with $i < j$. In this case, the computation $D_{i,j}$ did not involve any rotations by $O(n,\mathbb{R})$, so the composition of the $D_{i,j}$ in this case is well-defined (as a function on $\mathfrak{h}^n$ as a coset, not on $\mathrm{GL}(n,\mathbb{R})/(O(n,\mathbb{R}) \times \mathbb{R}^*)$ as a quotient). Our previous computations then show that the power function will be an eigenfunction, as desired. ☐

9.2 Langlands Eisenstein series for $\mathrm{SL}(n,\mathbb{Z})$

Recall that given a partition $n = n_1 + \cdots + n_r$, the associated **parabolic subgroup** is

$$\mathcal{P}_{n_1,\ldots,n_r} := \left\{ \begin{pmatrix} \mathrm{GL}(n_1) & & & * \\ & \mathrm{GL}(n_2) & & \\ & & \ddots & \\ & & & \mathrm{GL}(n_r) \end{pmatrix} \right\} \subseteq \mathrm{GL}(n,\mathbb{R}).$$

Given $s = (s_1, \ldots, s_r) \in \mathbb{C}^r$ such that $\sum_{i=1}^r n_i s_i = 0$, and

$$g = \begin{pmatrix} m_1(g) & & & * \\ & m_2(g) & & \\ & & \ddots & \\ & & & m_r(g) \end{pmatrix} \in \mathcal{P}_{n_1,\ldots,n_r},$$

with $m_i(g) \in \mathrm{GL}(n_i,\mathbb{R})$, we define the **power function**

$$|g|^s_{\mathcal{P}_{n_1,\ldots,n_r}} = \prod_{i=1}^r |\det(m_i)|^{s_i}.$$

Remark 9.2. *In the case of the Borel subgroup $n = 1 + 1 + \cdots + 1$, this agrees with the previous power function.*

For each $\mathrm{GL}(n_i)$, let $\phi_i : \mathfrak{h}^{n_i} \to \mathbb{C}$ be a Maass form, invariant by $\mathrm{SL}(n_i, \mathbb{Z})$. For $GL(1)$, we make the convention that $\phi_i = 1$.

Definition 9.3. *We have the* ***induced Maass form***

$$\Phi(g) := \prod_{i=1}^{r} \phi_i(m_i(g)).$$

Remark 9.4. *This will lead to the concept of parabolic induction.*

Note that $\Phi(ugkz) = \Phi(g)$ for all $g \in \mathcal{P}_{n_1,\ldots,n_r}$. $u \in N^{\mathcal{P}} = \left\{\begin{pmatrix} I_{n_1} & & & * \\ & I_{n_2} & & \\ & & \ddots & \\ & & & I_{n_r} \end{pmatrix}\right\}$, $k \in K = \left\{\begin{pmatrix} O(n_1,\mathbb{R}) & & & \\ & O(n_2,\mathbb{R}) & & \\ & & \ddots & \\ & & & O(n_r,\mathbb{R}) \end{pmatrix}\right\}$, and by $z \in Z$.

Definition 9.5. *Given parabolic subgroup $\mathcal{P} = \mathcal{P}_{n_1,\ldots,n_r}$ and induced Maass form Φ, we have the* ***Langlands Eisenstein series***

$$E_{\mathcal{P},\Phi}(g,s) := \sum_{\gamma \in (\Gamma \cap \mathcal{P})\backslash \Gamma} \Phi(\gamma g) |\gamma g|_{\mathcal{P}}^{s+\rho},$$

where $\rho(j) = \frac{n-n_j}{2} - n_1 - n_2 - \cdots - n_{j-1}$.

For today, we'll just consider some examples.

Example 9.6. *Let's consider the case of the partition $2 = 1 + 1$. We have*

- $s = (s_1, s_2)$*, with* $s_1 + s_2 = 0$.
- $\rho = (1/2, -1/2)$.
- $\Phi = 1$.
- $g = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$. *Using that the power function is invariant by K and Z, we can take* $g = \begin{pmatrix} y & x \\ & 1 \end{pmatrix}$.

Hence we get the Eisenstein series

$$E_{\mathcal{P}_{1,1}}(g,s) = \sum_{\gamma \in \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix} \backslash SL(2,\mathbb{Z})} |\gamma g|_{\mathcal{P}_{1,1}}^{s+\rho} = \sum_{\gamma \in \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix} \backslash SL(2,\mathbb{Z})} \left| \gamma \begin{pmatrix} y & x \\ & 1 \end{pmatrix} \right|_{\mathcal{P}_{1,1}}^{s_1+1/2} = \sum_{(c,d)=1} \frac{y^{s+1/2}}{|(cx+d)+icy|^{2s+1}}.$$

Example 9.7. *Now consider $3 = 2 + 1$. We have*

- $s = (s_1, s_2)$*, with* $2s_1 + s_2 = 0$.
- $\rho = (1/2, -1)$.
- *Let $\phi_1 : \mathfrak{h}^2 \to \mathbb{C}$ be a $SL(2,\mathbb{Z})$ Maass form, and $\phi_2 = 1$. Then $\Phi = \phi_1$.*
- $g = \begin{pmatrix} * & * & * \\ * & * & * \\ & & * \end{pmatrix} = \begin{pmatrix} m_1(g) & * \\ & m_2(g) \end{pmatrix} \in \mathcal{P}_{2,1}$*, where $m_1(g) \in GL(2,\mathbb{R})$ and $m_2(g) \in GL(1,\mathbb{R})$.*

Now

$$E_{\mathcal{P}_{2,1},\Phi}(g,s) = \sum_{\gamma \in \left(SL(3,\mathbb{Z}) \cap \begin{pmatrix} * & * & * \\ * & * & * \\ & & * \end{pmatrix} \right) \backslash SL(3,\mathbb{Z})} \Phi(m_1(\gamma g)))|\gamma g|_{\mathcal{P}2,1}^{s+\rho}.$$

Example 9.8. *For $4 = 2+2$, we have*

- $s = (s_1, s_2)$ *with* $2s_1 + 2s_2 = 0$
- $\rho = (1, -1)$
- *Let* $\phi_1 : \mathfrak{h}^2 \to \mathbb{C}$ *and* $\phi_2 : \mathfrak{h}^2 \to \mathbb{C}$ *be* $SL(2,\mathbb{Z})$ *Maass forms.*
- $g = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & * & * \end{pmatrix} = \begin{pmatrix} m_1(g) & * \\ & m_2(g) \end{pmatrix} \in \mathcal{P}_{2,2}$, *where* $m_1(g), m_2(g) \in GL(2,\mathbb{R})$.

Now

$$E_{\mathcal{P}_{2,2},\Phi}(g,s) = \sum_{\gamma \in \left(SL(4,\mathbb{Z}) \cap \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & * & * \end{pmatrix} \right) \backslash SL(4,\mathbb{Z})} \Phi_1(m_1(\gamma g))\Phi_2(m_2(\gamma g))|\gamma g|_{\mathcal{P}_{2,2}}^{s+\rho}.$$

Now, we will preview what we will talk about in the next few lectures.
For SL$(2,\mathbb{Z})$ Eisenstein series, we have a Fourier expansion: for $s = (s_1, -s_1)$, we have

$$E_{\mathcal{P}_{1,1}}(g,s) = y^{1/2+s_1} + \frac{\zeta^*(2s_1)}{\zeta^*(2s_1+1)} y^{1/2-s_1} + \frac{1}{\zeta^*(2s_1+1)} \sum_{m \neq 0} \sigma_{2s_1}(m)|m|^{-s_1}\sqrt{y} \int_0^\infty e^{-\pi|m|y(u+1/u)} u^{s_1} \frac{\mathrm{d}u}{u} \cdot e^{2\pi i m x},$$

where

$$\zeta^*(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s) = \zeta^*(1-s)$$

is the completed Riemann zeta function.
The function as is has poles because of the zeta function. To complete it, consider

$$\zeta^*(2s_1+1)E_{\mathcal{P}_{1,1}}(g,s);$$

This has functional equation $s = (s_1, s_2) \to (s_2, s_1)$.
The same thing happens with general Langlands functions as well – will need to multiply by the right thing to get the functional equation. In the $4 = 2+2$ case, the function needed to multiply is the Rankin-Selberg convolution $L(s, \phi_1 \otimes \phi_2)$; hence we can use this theory to get a functional equation for Rankin-Selberg L-functions.

Remark 9.9. *Is it possible to get something similar for Rankin-Selberg products of 3 or more Maass forms? Not on* $SL(n,\mathbb{Z})$*, but people are looking at Eisenstein series associated to other groups (loop groups?).*

10 Lecture 10 - 2/27/25

Last time, we talked about the Langlands Eisenstein series. This time, we want to compute the Fourier coefficients of the Eisenstein series.
There are two approaches:

- Explictily computing the Fourier coefficients via an integral. Generalized by Langlands to all $\mathrm{SL}(n, \mathbb{Z})$, but fairly annoying.
- Use Hecke operators. This approaches generalizes to $\mathrm{SL}(n, \mathbb{Z})$ more easily.

We motivate these approaches by looking at $\mathrm{SL}(2, \mathbb{Z})$.

10.1 Fourier Expansion for $\mathrm{SL}(2, \mathbb{Z})$ Eisenstein Forms

Consider an automorphic form F for $\mathrm{SL}(2, \mathbb{Z})$ (a term we haven't defined, but for now think of a Maass form or an Eisenstein series) with Langlands parameter $\alpha = (\alpha_1, \alpha_2)$. We previously described their Fourier expansion

$$F(z) = a_0 y^{1/2+\alpha_1} + a_0' y^{1/2+\alpha_2} + \sum_{n\neq 0} a_n \sqrt{y} K_{\alpha_1}(2\pi|n|y) e^{2\pi i n x},$$

where

$$K_\alpha(y) = \frac{1}{2}\int_0^\infty e^{-\frac{y}{2}(u+1/u)} u^\alpha \frac{\mathrm{d}u}{u}$$

is the Bessel function and the a_i are constants in $\mathbb{C}$. We normalize this function by letting $a_1 = 1$.
Recall from last time that we have the classical Eisenstein series

$$E_{\mathcal{P}_{1,1}}(z, s) = \sum_{\gamma \in \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix} \backslash \mathrm{SL}(2,\mathbb{Z})} (\operatorname{Im} \gamma z)^{s+1/2} = \sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix} \backslash \mathrm{SL}(2,\mathbb{Z})} \left(\frac{y}{|cz+d|^2}\right)^{s+1/2}.$$

We wish to compute the Fourier expansion. How do we do so? Langlands computed the Fourier expansion by computing the integral

$$\begin{aligned}
\zeta(2s+1)\int_0^1 E_{\mathcal{P}_{1,1}}(x+iy, s) e^{-2\pi i n x}\,\mathrm{d}x &= \int_0^1 \zeta(2s+1) \sum_{(c,d)=1} \frac{y^{s+1/2}}{((cx+d)^2+(cy)^2)^{s+1/2}} e^{-2\pi i n x}\,\mathrm{d}x \\
&= \int_0^1 \sum_{(c,d)\neq(0,0)} \frac{y^{s+1/2}}{((cx+d)^2+(cy)^2)^{s+1/2}} e^{-2\pi i n x}\,\mathrm{d}x \\
&= 2\zeta(2s+1) y^{s+1/2}\delta_{n=0} + \int_0^1 \sum_{c\neq 0} \frac{y^{s+1/2}}{((cx+d)^2+(cy)^2)^{s+1/2}} e^{-2\pi i n x}\,\mathrm{d}x
\end{aligned}$$

where the first term corresponds to the $c = 0$ term. To compute the second integral, we note that

$$\begin{aligned}
\int_0^1 \sum_{c\neq 0} \frac{y^{s+1/2}}{((cx+d)^2+(cy)^2)^{s+1/2}} e^{-2\pi i n x}\,\mathrm{d}x &= \sum_{c\neq 0}\sum_{m=-\infty}^{\infty}\sum_{r=1}^{c}\int_0^1 \frac{y^{s+1/2}}{((cx+mc+r)^2+(cy)^2)^{s+1/2}} e^{-2\pi i n x}\,\mathrm{d}x \\
&= \sum_{c\neq 0}\frac{1}{|c|^{2s+1}}\sum_{m=-\infty}^{\infty}\sum_{r=1}^{|c|}\int_{-m-r/c}^{1-m-r/c} \frac{y^{s+1/2}}{(x^2+y^2)^{s+1/2}} e^{-2\pi i n(x-r/c)}\,\mathrm{d}x \\
&= 2\sum_{c=1}^{\infty}\frac{1}{c^{2s+1}}\sum_{r=1}^{c} e^{-2\pi i r/c}\int_{-\infty}^{\infty} \frac{y^{s+1/2}}{(x^2+y^2)^{s+1/2}} e^{-2\pi i n x}\,\mathrm{d}x
\end{aligned}$$

where we apply the transformation $x \to x - m - 1/c$, then

$$
\begin{aligned}
=&2\sum_{c=1}^{\infty}\frac{1}{c^{2s+1}}\sum_{r=1}^{c}e^{-2\pi i r/c}\int_{-\infty}^{\infty}\frac{y^{s+1/2}}{y^{2s+1}(x^2+1)^{s+1/2}}e^{-2\pi i n x y}y\,\mathrm{d}x\\
=&2\sum_{c|n}c^{-2s}y^{1/2-s}\int_{-\infty}^{\infty}\frac{1}{(x^2+1)^{s+1/2}}e^{-2\pi i n x y}\,\mathrm{d}x\\
=&\begin{cases}2\zeta(2s)y^{1/2-s}\sqrt{\pi}\frac{\Gamma(s)}{\Gamma(s+1/2)} & n=0\\ 4\sigma_{-2s}(n)y^{1/2}\frac{\pi^{s+1/2}|n|^s}{\Gamma(s+1/2)}K_s(2\pi n|y|) & n\neq 0\end{cases}
\end{aligned}
$$

where here we apply the transformation $x \to xy$, and the last line is a known integral computation. Here $\sigma_k(n) = \sum_{d|n} n^k$.
Hence we get that the Fourier expansion is

$$E_{\mathcal{P}_{1,1}}(z,s) = 2y^{s+1/2} + 2y^{1/2-s}\frac{\sqrt{\pi}\Gamma(s)\zeta(2s)}{\Gamma(s+1/2)\zeta(2s+1)} + \frac{4\pi^{s+1/2}y^{1/2}}{\Gamma(s+1/2)\zeta(2s+1)}\sum_{n\neq 0}\sigma_{-2s}(n)|n|^s K_s(2\pi n|y|)e^{2\pi i n x}$$

10.2 Hecke Operators for SL$(2,\mathbb{Z})$

For $\mathrm{SL}(2,\mathbb{Z})$, let $F : \mathrm{SL}(2,\mathbb{Z})\backslash\mathfrak{h}^2 \to \mathbb{C}$ be an automorphic form. For $n = 1, 2, \ldots$, the Hecke operators are defined by

$$T_n F(z) = \frac{1}{\sqrt{n}}\sum_{\substack{ad=n\\0\le b<d}} F\left(\frac{az+b}{d}\right).$$

Theorem 10.1 (Hecke)**.** *$T_n F(z) = a_n F(z)$.*

The L-function associated with F is

$$L(s,f) := \sum_{n=1}^{\infty}\frac{a_n}{n^s}.$$

This has an Euler product

$$\prod_p\prod_{i=1}^{2}\left(1-\frac{\alpha_i(p)}{p^s}\right)^{-1},$$

where the $\alpha_i(p)$ are the roots of a quadratic involving a_p.

Remark 10.2. *The Hecke operators are Γ-invariant, and adelically they act at the finite places (in comparison to the differential operators, which act at the infinite place).*

Now with the Hecke operators, we can define the concept of an automorphic form:

Definition 10.3. *An **automorphic form for SL$(2,\mathbb{Z})$ with Langlands parameters** $\alpha = (\alpha_1, \alpha_2)$ is a smooth function $F : \mathfrak{h}^2 \to \mathbb{C}$ such that*

- *$F(\gamma z) = F(z)$ for all $\gamma \in SL(2,\mathbb{Z})$*
- *$F(z) \ll y^B$ for $z = x+iy$ and some $B > 0$ fixed.*
- *$\Delta F = \left(\frac{1}{4} - \alpha_1^2\right)F$*
- *$T_n F = a_n F$.*

To compute T_n on $E(z,s)$, it is enough to compute T_n on the power function, since T_n is $\mathrm{SL}(2,\mathbb{Z})$-invariant. In particular, note that

$$T_n(\operatorname{Im} z)^{s+1/2} = \frac{1}{\sqrt{n}}\sum_{\substack{ad=n\\0\le b<d}}\left(\operatorname{Im}\frac{az+b}{d}\right)^{s+1/2} = \frac{1}{\sqrt{n}}\sum_{\substack{ad=n\\0\le b<d}}\left(\frac{ny}{d^2}\right)^{s+1/2} = n^s\sigma_{-2s}(n)(\operatorname{Im} z)^{s+1/2},$$

and $n^s\sigma_{-2s}(n)$ appears in the expansion of the Eisenstein series.
Thus, in general, to compute the Fourier expansion for $\mathrm{SL}(n,\mathbb{Z})$ automorphic forms, we will need to extend the theory of Hecke operators.

10.3 Hecke Operators in General

Let G be a group, acting on a topological space X. Let $\Gamma \subseteq G$ be a discrete subgroup.

Definition 10.4. *An element $g \in G$ is a **commensurator** if $(g^{-1}\Gamma g) \cap \Gamma$ is of finite index in $g^{-1}\Gamma g$ and Γ.*

We have the **commensurator subgroup** $C_G(\Gamma) = \left\{g \in G \mid (g^{-1}\Gamma g) \cap \Gamma \text{ finite index in } g^{-1}\Gamma g \text{ and } \Gamma\right\}$. Consider $g \in C_G(\Gamma)$ and letting d be the index of $(g^{-1}\Gamma g) \cap \Gamma$ in Γ, we have a right coset decomposition

$$\Gamma = \cup_{i=1}^d ((g^{-1}\Gamma g) \cap \Gamma)\delta_i,$$

with the $\delta_i \in \Gamma$. Equivalently, this can be written as

$$\Gamma g \Gamma = \cup_{i=1}^d \Gamma g \delta_i.$$

Definition 10.5 (Hecke Operator)**.** *Consider $F \in \mathcal{L}^2(\Gamma\backslash X)$ and a fixed $g \in C_G(\Gamma)$. We define*

$$T_g F(x) := \sum_{i=1}^d F(g\delta_i x).$$

Note that the definition is independent of choice of δ_i because if δ_i and δ_i^* are in the same coset, then $\delta_i = g^{-1}\gamma g \delta_i^*$, or equivalently $g\delta_i = \gamma g \delta_i^*$ for some $\gamma \in \Gamma$, and F is Γ left-invariant.

Claim 10.6. $T_g : \mathcal{L}^2(\Gamma\backslash X) \to \mathcal{L}^2(\Gamma\backslash X)$.

The $\mathcal{L}^2$ condition is simple to check; this is really checking that

$$(T_g F)(\gamma x) = (T_g F)(x)$$

for all $\gamma \in \Gamma$.

Proof. We have by definition

$$T_g F(\delta x) = \sum_{i=1}^d F(g\delta_i \gamma x).$$

By the coset decomposition, we can write $\delta_i \gamma = \delta_i' \delta_{\sigma(i)}$, where $\sigma \in S_d$ is a permutation of $1, \ldots, d$ and $\delta_i' \in (g^{-1}\Gamma g) \cap \Gamma$. Moreover, $g\delta_i\gamma = g\delta_i'\delta_{\sigma(i)} = \delta_i'' g \delta_{\sigma(i)}$ for some $\delta_i'' \in \Gamma$. Thus

$$T_g F(\gamma x) = \sum_{i=1}^d F(\delta_i'' g \delta_{\sigma(i)} x) = \sum_{i=1}^d F(g\delta_{\sigma(i)} x) = T_g F(x).$$

□

We have a additive group of Hecke operators by taking the additive group generated by the T_g.

Example 10.7. *For $\Gamma = SL(2,\mathbb{Z})$, $G = GL(2,\mathbb{R})$, $X = \mathfrak{h}^2$, and $F \in \mathcal{L}^2(\Gamma\backslash\mathfrak{h}^2)$. Note that the element*

$$g = \begin{pmatrix} n_0 n_1 & 0 \\ 0 & n_0 \end{pmatrix} \in C_G(\Gamma),$$

with $n_0, n_1 \in \mathbb{Z}$. Consider

$$S_n = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mid ad = n, 0 \le b < d \right\}.$$

By taking our Hecke operator to be the sum of T_g, with $g = \begin{pmatrix} n_0 n_1 & 0 \\ 0 & n_0 \end{pmatrix}$ with determinant n, one can check that the elements of S_n can be used as the δ_i in our decomposition. This returns the original Hecke operator we defined earlier.

Theorem 10.8. *For* $SL(2,\mathbb{Z})$, $T_{mn} = T_m T_n$ *if* $(m,n) = 1$.

To prove this in general, we need to find a way to muliply Hecke operators and express the answer as a sum of Hecke operators; i.e. we want to extend our additive group of Hecke operators to a ring.
In particular, consider $g, h \in C_G(\Gamma)$, with $\Gamma g \Gamma = \cup_i \Gamma \alpha_i$ and $\Gamma g \Gamma = \cup_i \Gamma \beta_i$. We have

$$(\Gamma g \Gamma)(\Gamma h \Gamma) = (\Gamma g \Gamma) \cup_j \Gamma \beta_j = \cup_{i,j} \Gamma \alpha_i \beta_j.$$

Then the product $T_g T_h$ corresponds to summing over the $\alpha_i \beta_j$.
Next time, we will prove that the Hecke operators in general are commutative, then talk about the $\mathrm{GL}(n)$ Hecke operators.

11 Lecture 11 - 3/4/25

11.1 Hermite and Smith Normal Form

We take a small aside to talk about Hermite and Smith normal forms for integer matrices, which we will use in the discussion for Hecke operators.
Let $A \in \mathrm{GL}(n, \mathbb{Z})^+$. Then there exists $\gamma \in \mathrm{SL}(n, \mathbb{Z})$ such that

$$\gamma A = \begin{pmatrix} c_1 & c_{1,2} & \cdots & c_{1,n} \\ & c_2 & \cdots & c_{2,n} \\ & & \ddots & \vdots \\ & & & c_n \end{pmatrix}$$

where the $c_i \geq 1$ and the $c_{i,\ell} < c_\ell$. This is called the **Hermite normal form**. This exists because left multiplication by $\mathrm{SL}(n, \mathbb{Z})$ corresponds to row operations on the matrix.
Similarly, there exists $\gamma, \gamma' \in \mathrm{SL}(n, \mathbb{Z})$ such that

$$\gamma A \gamma' = \begin{pmatrix} d_n & & & \\ & \ddots & & \\ & & d_2 & \\ & & & d_1 \end{pmatrix}$$

with $d_i > 0$, and $d_1 \mid d_2 \mid \cdots \mid d_n$. This is called the **Smith normal form**, which exists because now we also have column operations on the matrix.
The uniqueness of these forms can be proven directly by comparing what $\mathrm{SL}(n, \mathbb{Z})$ matrix would be needed to convert from one form to the other.
We introduce these forms to discuss Hecke operators for $\mathrm{SL}(n, \mathbb{Z})$.

11.2 Hecke operators in general

Consider a group G acting on topological space X, with discrete subgroup Γ. We have the commensurator subgroup

$$C_G(\Gamma) = \left\{ g \in G \mid (g^{-1}\Gamma g) \cap \Gamma \text{ has finite index in } \Gamma \text{ and } g^{-1}\Gamma g \right\}.$$

For any $g \in \Gamma$, we have double coset decomposition

$$\Gamma g \Gamma = \cup_{i=1}^{d} \Gamma \alpha_i.$$

With this decomposition, we define the associated Hecke operator $T_g : \mathcal{L}^2(\Gamma \backslash X) \to \mathcal{L}^2(\Gamma \backslash X)$ by

$$T_g F(x) := \sum_{i=1}^{d} F(\alpha_i x).$$

In particular, we showed last time that $T_g F(\gamma x) = T_g F(x)$.
Define an **antiautomorphism** on a group G to be a map $* : G \to G$ such that $(x_1 x_2)^* = x_2^* x_1^*$. In the case of matrix groups, the matrix transpose is an antiautomorphism.

Lemma 11.1. *If there exists $*$ such that $\Gamma^* = \Gamma$ and $(\Gamma g \Gamma)^* = \Gamma g \Gamma$, then $T_{gh} = T_g T_h$; i.e. the Hecke operators commute.*

The proof can be found in Dorian's book.

11.3 Hecke operators for $\mathbf{SL}(n,\mathbb{Z})$

We now specialize to the case $G = \mathrm{GL}(n,\mathbb{R})$, $\Gamma \in \mathrm{SL}(n,\mathbb{Z})$ and $X = \mathfrak{h}^n = \mathrm{GL}(n,\mathbb{R})/(O(n,\mathbb{R}) \cdot \mathbb{R}^*)$.
By the previous discussion, the Hecke operators on $\mathrm{SL}(n,\mathbb{Z})$ commute. Moreover, it is easy to verify that the Hecke operators commute with all of the left $\mathrm{GL}(n,\mathbb{R})$-invariant differential operators. By functional analysis, we can decompose $\mathcal{L}^2(\Gamma\backslash\mathfrak{h}^n)$ into a basis of simultaneous eigenfunctions of all the differential operators and Hecke operators. (There is a spectral theorem for unbounded operators, which will not talk about.)
Such a function will be called a **Hecke form** - a (smooth) Maass form that is simultaneous eigenfunctions of the differential operators.
We now construct the Hecke operators for $\mathrm{SL}(n,\mathbb{Z})$. For any positive integer m, consider the set D_m of all matrices of the form

$$m' = \begin{pmatrix} m_0 m_1 \dots m_{n-1} & & & \\ & \ddots & & \\ & & m_0 m_1 & \\ & & & m_0 \end{pmatrix}$$

of determinant m. We want to compute T_m, the Hecke operator corresponding to the union of all of the matrices in D_m. In particular, this will mean looking at the α_i in the double coset decomposition $\bigcup_{m' \in D_m} \Gamma m' \Gamma = \bigcup \Gamma \alpha_i$.
Let

$$S_m = \left\{ \begin{pmatrix} c_1 & c_{1,2} & \cdots & c_{1,n} \\ & c_2 & \cdots & c_{2,n} \\ & & \ddots & \vdots \\ & & & c_n \end{pmatrix} \mid c_i \geq 1, 0 \leq c_{i,\ell} < c_\ell, c_1 c_2 \dots c_n = \det(m) \right\}.$$

Claim 11.2.

$$\bigcup_{m' \in D_m} \Gamma m' \Gamma = \bigcup_{\substack{c_i \geq 1 \\ 0 \leq c_{i,\ell} < c_\ell \\ c_1 c_2 \dots c_n = \det(m)}} \Gamma \begin{pmatrix} c_1 & c_{1,2} & \cdots & c_{1,n} \\ & c_2 & \cdots & c_{2,n} \\ & & \ddots & \vdots \\ & & & c_n \end{pmatrix}.$$

The proof of this claim follows directly from the uniqueness of Hermite and Smith normal forms.
Hence we have the Hecke operator

$$T_m F(g) = \sum F\left(\begin{pmatrix} c_1 & c_{1,2} & \cdots & c_{1,n} \\ & c_2 & \cdots & c_{2,n} \\ & & \ddots & \vdots \\ & & & c_n \end{pmatrix} g \right) = \lambda_m F(g)$$

for F a Hecke form.

11.4 Hecke operators applied to Borel Eisenstein series

Consider the Borel subgroup B of upper triangular matrices (the parabolic subgroup corresponding to $n = 1 + 1 + \cdots + 1$) and $b \in B$, with Y_i the elements on the diagonal. We have the power function

$$|b|_B^s = \prod_{i=1}^{n} Y_i^{s_i},$$

where $s = (s_1, s_2, \dots, s_n)$ with $s_1 + \cdots + s_n = 0$. This function satsifies

$$|ubkz|_B^s = |b|_B^s,$$

where u is unipotent, $k \in K = O(n,\mathbb{R})$, and $z \in Z$ is in the center of $\mathrm{GL}(n,\mathbb{R})$.

Then
$$E_B(g,s) = \sum_{\gamma \in \Gamma\cap B\backslash\Gamma} |\gamma g|_B^{s+\rho},$$
where $\rho_i = \frac{n+1}{2} - i$.
To compute $T_m E_B(g,s)$, it is enough to compute $T_m|g|_B^{s+\rho}$. In particular,

$$\begin{aligned}
T_m|g|_B^{s+\rho} &= \sum_{\substack{c_i\geq 1\\ 0\leq c_{i,\ell}<c_\ell \\ c_1c_2\ldots c_n=\det(m)}} \left|\begin{pmatrix} c_1 & c_{1,2} & \cdots & c_{1,n}\\ & c_2 & \cdots & c_{2,n}\\ & & \ddots & \vdots\\ & & & c_n\end{pmatrix} g\right|_B^{s+\rho}\\
&= \sum_{\substack{c_i\geq 1\\ 0\leq c_{i,\ell}<c_\ell \\ c_1c_2\ldots c_n=\det(m)}} \left|\begin{pmatrix} c_1 & & & \\ & c_2 & & \\ & & \ddots & \\ & & & c_n\end{pmatrix}\begin{pmatrix} y_1\cdots y_{n-1} & & & \\ & y_1\cdots y_{n-2} & & \\ & & \ddots & \\ & & & 1\end{pmatrix}\right|_B^{s+\rho}\\
&= \sum_{\substack{c_i\geq 1\\ 0\leq c_{i,\ell}<c_\ell \\ c_1c_2\ldots c_n=\det(m)}} \prod_{i=1}^n (c_i y_1\ldots y_{n-i})^{s_i+\rho_i}\\
&= \left(\sum_{\substack{c_i\geq 1\\ 0\leq c_{i,\ell}<c_\ell \\ c_1c_2\ldots c_n=\det(m)}} \prod_{i=1}^n c_i^{s_i+\rho_i}\right)|g|_B^{s+\rho},
\end{aligned}$$

which is a divisor sum. We will show that this correspond to the Fourier coefficient of the mth term of the Eisenstein series.
Next time, we will do this for Eisenstein series for an arbitrary parabolic. This time, the c_i will be replaced by Fourier coefficients of the Maass forms that are induced.

12 Lecture 12 - 3/6/25

Up to now we've been focusing on Eisenstein series. Let's switch back to Maass forms.

12.1 Maass Forms and Whittaker Functions for SL$(n, \mathbb{Z})$

Recall:

Definition 12.1. *A* ***SL***$(n,\mathbb{Z})$ ***(Hecke-)Maass form*** ϕ *is a smooth function* $\phi : \mathfrak{h}^n \to \mathbb{C}$ *such that*

- $\phi(\gamma g) = \phi(g)$ *for all* $\gamma \in \Gamma_n$, $g \in \mathfrak{h}^n$
- $D\phi = \lambda_D \phi$ *for all* $D \in ZU(\mathfrak{gl}(n,\mathbb{R}))$, *where* λ_D *matches the eigenvalue of the power function.*
- $T_N\phi = \lambda'_N \phi$ *for all Hecke operators*

$$T_N\phi = \frac{1}{N^{\frac{n-1}{2}}} \sum_{\substack{c_i \geq 1 \\ 0 \leq c_{i,\ell} < c_\ell \\ c_1 c_2 \ldots c_n = N}} \phi\left(\begin{pmatrix} c_1 & c_{1,2} & \cdots & c_{1,n} \\ & c_2 & \cdots & c_{2,n} \\ & & \ddots & \vdots \\ & & & c_n \end{pmatrix} g\right)$$

- $\int_{\Gamma_n \backslash \mathfrak{h}^n} |\phi(g)|^2 \, \mathrm{d}g < \infty$ *(moderate growth)*

Here we are defining a Maass form at the archimedean place. It is possible to lift this to a Maass form adelically, but we will not talk about this in the class. (For the case of SL$(n,\mathbb{Z})$, the lift gives nothing new.)

Consider the group $U_n(\mathbb{R}) = \left\{ \begin{pmatrix} 1 & & u_{ij} \\ & \ddots & \\ & & 1 \end{pmatrix} \right\}$, and let $M = (m_1, m_2, \ldots, m_{n-1}) \in \mathbb{Z}^{n-1}$. We have a character ψ_M on $U_n(\mathbb{R})$, defined by

$$\psi_M(u) = e^{2\pi i (m_1 u_{1,2} + m_2 u_{2,3} + \cdots + m_{n-1} u_{n-1,n})}.$$

In particular, it is easy to check that $\psi_M(uu') = \psi_M(u)\psi_M(u')$.

Example 12.2. *Let* $n = 3$, $M = (m_1, m_2)$, *and* $u = \begin{pmatrix} 1 & u_1 & u_3 \\ & 1 & u_2 \\ & & 1 \end{pmatrix}$. *Then* $\psi_M(u) = e^{2\pi i (m_1 u_1 + m_2 u_2)}$.

We want to use these characters to give the Fourier expansion of the Maass forms. However, since $U_n(\mathbb{R})$ is not commutative, this is more difficult.
The analogue of a Fourier coefficient of ϕ in this case is

$$\widehat{\phi}_M(g) = \int_0^1 \cdots \int_0^1 \phi(ug)\overline{\psi_M(u)} \, \mathrm{d}u \, ,$$

where $\mathrm{d}u = \prod_{1 \leq i < j \leq n} \mathrm{d}u_{i,j}$. This will inherit properties of the Maass forms. In particular,

- $\widehat{\phi}_M(gkz) = \widehat{\phi}_M(g)$ for all $k \in K_n = O(n,\mathbb{R})$, and $z \in Z$.
- $\widehat{\phi}_M(vg) = \psi_M(v)\widehat{\phi}_M(g)$
- $D\widehat{\phi}_M(g) = \lambda_D \widehat{\phi}_M(g)$
- $\int_{\Gamma_n \backslash \mathfrak{h}^n} |\widehat{\phi}_M(g)|^2 \, \mathrm{d}g < \infty$

How many functions satisfy all these properties? Shalika showed (Mulitplicity One theorem) that there is only one up to constant multiple. The proof for SL$(2,\mathbb{Z})$ is simple, and there is a proof for SL$(3,\mathbb{Z})$ in Dorian's book, but there is no known (at least to Dorian) simple proof for SL$(n,\mathbb{Z})$.

Remark 12.3. *When working adelically, we need local Whittaker functions. Multiplicity one was proved for them.*

Jacquet constructed the function satsifying all of these properties. Recall we have the power function $|g|^s$ (here $s = (s_1, \ldots, s_n) = \alpha + \rho$, with $\sum s_i = 0$). Recall that

$$|g|^s = (y_1 \ldots y_{n-1})^{s_1}(y_1 y_2 \ldots y_{n-2})^{s_2} \ldots y_1^{s_{n-1}}.$$

Jacquet constructed the **Jacquet-Whittaker function**

$$W_M(g) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |w_n u g|^s \overline{\psi_M}(u) \, \mathrm{d}u \,,$$

where $\mathrm{d}u = \prod_{1 \le i < j \le n} \mathrm{d}u_{i,j}$ and $w_n = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{pmatrix}$ is the long element of the Weyl group. Here the long element is needed for convergence, and this integral will converge for $\mathrm{Re}(s_i)$ sufficiently large.
In particular, multiplicity one gives that $\widehat{\phi}_M(g) = c_M W_M(g)$. What is this constant? It will turn out to essentially be the Hecke eigenvalue.

12.2 Applications

We have the Whittaker expansion (for even Maass form ϕ)

$$\phi(g) = \sum_{\gamma \in U_{n-1} \backslash \Gamma_{n-1}} \sum_{m_1=1}^{\infty} \cdots \sum_{m_{n-1}=1}^{\infty} \frac{A(m_1, \ldots, m_{n-1})}{\prod_{i=1}^{n-1} m_k^{k(n-k)/2}} W_M \left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g \right).$$

(Here even means that ϕ is $\phi(g) = \phi(\delta g)$, where

$$\delta = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}.$$

)
Then we can show that there exists an L-function

$$L(s, \phi) = \sum_{m=1}^{\infty} A(m, 1 \ldots, 1) m^{-s}$$

with a functional equation, which will be similar to the Riemann zeta function (n Gamma factors instead of 1).
This also has an Euler product

$$\prod_p \left(1 - A(p, 1, \ldots, 1)p^{-s} + \cdots + (-1)^{n-1} A(1, \ldots, 1, p) p^{-(n-1)s} + (-1)^n p^{-ns}\right)^{-1}.$$

Example 12.4. *Let $n = 3$, $M = (m_1, m_2)$, and $u = \begin{pmatrix} 1 & u_1 & u_3 \\ & 1 & u_2 \\ & & 1 \end{pmatrix}$, $M^* = \begin{pmatrix} m_1 m_2 & & \\ & m_2 & \\ & & 1 \end{pmatrix}$, $g = xy$, and $w_3 = \begin{pmatrix} & & -1 \\ & 1 & \\ 1 & & \end{pmatrix}$ (the definition is equivalent with -1 instead of 1 in the top right). Then the Whittaker function is*

$$W_n(g) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |w_3 u g|^s e^{-2\pi i (m_1 u_1 + m_2 u_2)} \, \mathrm{d}u_1 \, \mathrm{d}u_2 \,.$$

Claim 12.5. *$W_M(g) = c_{s,M} W_{(1,1)}(M^*g)$, where the constant only depends on s and M.*

Proof. We have that

$$\begin{aligned}
W_{(1,1)}(M^*g) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |w_3 u M^* g|^s e^{-2\pi i(u_1+u_2)}\,\mathrm{d}u_1\,\mathrm{d}u_2\,\mathrm{d}u_3 \\
&= m_1 m_2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |w_3 M^* u g|^s e^{-2\pi i(m_1 u_1' + m_2 u_2')}\,\mathrm{d}u_1'\,\mathrm{d}u_2'\,\mathrm{d}u_3 \\
&= m_1 m_2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |w_3 M^* w_3^{-1}|^s |w_3 u g|^s e^{-2\pi i(m_1 u_1' + m_2 u_2')}\,\mathrm{d}u_1'\,\mathrm{d}u_2'\,\mathrm{d}u_3 \\
&= c_{s,M} W_{(m_1,m_2)}(g).
\end{aligned}$$

where $u_1 = u_1' m_1$ and $u_2 = u_2' m_2$. □

Remark 12.6. *This proof holds in general for all n; the constant will depend only on n, s, and M.*

Our next goal is to show that $T_N\phi(g) = A(N,1,\ldots,1)\phi(g)$. The Whittaker expansion gives that

$$\int_0^1 \cdots \int_0^1 \phi(ug)\overline{\psi_M(u)}\,\mathrm{d}u = \frac{A(m_1,\ldots,m_{n-1})}{\prod_{k=1}^n m_k^{k(n-k)/2}} W_M(g).$$

Applying T_N to $\widehat{\phi}_M(g)$ gives

$$\begin{aligned}
T_N\widehat{\phi}_M(g) &= \int_0^1 \cdots \int_0^1 \sum_{\substack{c_i\ge 1\\ 0\le c_{i,\ell}<c_\ell \\ c_1c_2\ldots c_n = N}} \phi\left(\begin{pmatrix} c_1 & c_{1,2} & \cdots & c_{1,n} \\ & c_2 & \cdots & c_{2,n} \\ & & \ddots & \vdots \\ & & & c_n \end{pmatrix} ug\right)\overline{\psi_M(u)}\,\mathrm{d}u \\
&= \lambda_N \int_0^1 \cdots \int_0^1 \phi(ug)\overline{\psi_M(u)}\,\mathrm{d}u\,.
\end{aligned}$$

using that $T_n\phi = \lambda_N\phi$.

Let $C^* = \begin{pmatrix} c_1 & c_{1,2} & \cdots & c_{1,n} \\ & c_2 & \cdots & c_{2,n} \\ & & \ddots & \vdots \\ & & & c_n \end{pmatrix}$. We can express $C^*u = u'C^*$, where

$$u'_{ij} = \frac{1}{c_j}\sum_{k=1}^{j} c_{i,k}u_{k,j}.$$

Making this change of variables gives that

$$T_N\widehat{\phi}_M(g) = \sum_{\substack{c_i\ge 1\\ 0\le c_{i,\ell}<c_\ell \\ c_1c_2\ldots c_n = N}} \prod_{1\le i<j\le n} \int_{u_{i,j}=*}^{*} \phi\left(u'\begin{pmatrix} c_1 & & \\ & \ddots & \\ & & c_n \end{pmatrix} g\right) e^{-2\pi i(*)}\,\mathrm{d}u'\,,$$

where the $*$ is a pretty bad computation (that can be found in Dorian's book). Working through it gives

$$\sum_{\substack{c_i\ge 1\\ 0\le c_{i,\ell}<c_\ell \\ c_1c_2\ldots c_n = N}} \prod_{1\le i<j\le n} \int_{u_{i,j}=0}^{\frac{c_iN}{c_j}} \phi\left(u'\begin{pmatrix} c_1 & & \\ & \ddots & \\ & & c_n \end{pmatrix} g\right) e^{-2\pi i \sum_{r=1}^{n-1} \frac{c_{r+1}m_{n-r}}{c_r} u'_{r,r+1}} \frac{c_j}{c_i}\,\mathrm{d}u'\,.$$

Working through the mess gives

$$\lambda_N A(m_1, \ldots, m_{n-1}) = \sum_{\substack{c_1 c_2 \ldots c_n = N \\ c_{n-1} | m_1, c_{n-2} | m_2, \ldots c_1 | m_{n-1}}} A\left(\frac{m_1 c_n}{c_{n-1}}, \frac{m_2 c_{n-1}}{c_{n-2}}, \ldots, \frac{m_{n-1} c_2}{c_1}\right).$$

Next time, we will use this relation to finish up the computation regarding Hecke operators.

13 Lecture 13 - 3/11/25

13.1 Fourier Coefficients of SL$(n,\mathbb{Z})$ Maass Forms

We briefly review what we discussed last time. Let F be an (even) Maass form for $\Gamma_n = \mathrm{SL}(n,\mathbb{Z})$ with Langlands parameters $\alpha = (\alpha_1,\dots,\alpha_n)$. Then we have a Fourier expansion

$$F(g) = \sum_{\gamma \in U_{n-1}\backslash \Gamma_{n-1}} \sum_{m_1=1}^{\infty} \cdots \sum_{m_{n-1}=1}^{\infty} \frac{A(m_1,\dots,m_{n-1})}{\prod_{k=1}^{n-1} m_k^{\frac{k(n-k)}{2}}} W\left(M\begin{pmatrix}\gamma & \\ & 1\end{pmatrix} g, \alpha, \psi_{1,\dots,1}\right),$$

where W is the Jacquet-Whittaker function

$$W(g,\alpha,\psi) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |ug|^{\alpha+\rho}\overline{\psi(u)}\, \mathrm{d}u$$

and

$$M = \begin{pmatrix} m_1 m_2 \dots m_{n-1} & & & \\ & m_2 \dots m_{n-1} & & \\ & & \ddots & \\ & & & m_{n-1} \end{pmatrix}$$

(Note that this matrix is opposite to Dorian's book, because in Dorian's book the elements above the diagonal are reversed in order.)
Last time, applying the Hecke operator T_N to this expansion gave the identity

$$\lambda_N A(m_1,\dots,m_{n-1}) = \sum_{\substack{c_1 c_2 \dots c_n = N \\ c_{n-1}|m_1, c_{n-2}|m_2, \dots c_1|m_{n-1}}} A\left(\frac{m_1 c_n}{c_{n-1}}, \frac{m_2 c_{n-1}}{c_{n-2}}, \dots, \frac{m_{n-1} c_2}{c_1}\right).$$

(More details can be found in Dorian's book.)

Proposition 13.1. *If $A(1,\dots,1) = 0$, then $A(m_1,\dots,m_{n-1}) = 0$ for all m_i. If $A(1,\dots,1) \neq 0$, normalize it to be 1. Then $\lambda_N = A(N,1,\dots,1)$. In particular, this gives the identity*

$$A(N,1,\dots,1)A(m_1,\dots,m_{n-1}) = \sum_{\substack{c_1 \dots c_n = N \\ c_1|m_1, \dots, c_{n-1}|m_{n-1}}} A\left(\frac{m_1 c_n}{c_1}, \frac{m_2 c_1}{c_2}, \dots, \frac{m_{n-1} c_{n-2}}{c_{n-1}}\right)$$

Proof. More details for this proof can be found in Dorian's book. First, one needs to prove a multiplicativity relation for relatively prime coefficients; see Dorian's paper.
By setting all of the $m_i = 1$, one can show that $A(1,\dots,1) = 0$ implies that $A(N,1,\dots,1) = 0$ for all N.
Next, set $N = m_1 = p$ and $m_2 = \dots = m_{n-1} = 1$. Then

$$0 = \lambda_p A(p,1,\dots,1) = \sum_{c_1 c_n = p} A\left(\frac{p^2}{c_1^2}, c_1, 1, \dots, 1\right),$$

so we conclude that $A(1,p,1,\dots,1) = 0$.
One can inductively take higher powers of p and later positions of $A(1,\dots,1,p,1\dots,1)$ (and apply the multiplicativity relation) prove the relation.
Now, suppose $A(1,\dots,1) = 1$. Setting $m_i = 1$ in the previous relation gives the final equation. □

13.2 L-functions of SL$(n,\mathbb{Z})$ Maass forms

To a Maass form F, we associate the L-function

$$L(s,F) = \sum_{n=1}^{\infty} \frac{A(m,1,\dots,1)}{m^s}.$$

We can show that this L-function satsifies an Euler product.

Proposition 13.2. *Define*

$$\phi_p(s) = \sum_{k=0}^{\infty} \frac{A(p^k, 1, \ldots, 1)}{p^{ks}}.$$

Then L-function has an Euler product

$$L(s, F) = \prod_p \phi_p(s),$$

where

$$\phi_p(s) = \Big(1 - A(p, 1, \ldots, 1)p^{-s} + A(1, p, 1, \ldots, 1)p^{-2s} + \cdots + (-1)^{n-1}A(1, \ldots, 1, p)p^{-(n-1)s} + (-1)^n p^{-ns}\Big)^{-1}.$$

Proof. For positive integer k, we have the relation

$$A(p^k, 1, \ldots, 1)A(p, 1, \ldots, 1) = A(p^{k+1}, 1, \ldots, 1) + A(p^{k-1}, p, 1, \ldots, 1) + \ldots.$$

Similarly,

$$A(p^k, 1, \ldots, 1)A(1, p, \ldots, 1) = A(p^k, p, \ldots, 1) + A(p^{k-1}, 1, p, \ldots, 1) + \ldots.$$

One can repeat this for all positions of the p. Adding them all up with alternative signs and multiplying by the right power of p^{-s} gives the desired result. More details can be found in Dorian's book. ☐

Example 13.3. *In the $SL(3, \mathbb{Z})$ case, we have the L-function*

$$L(s, F) = \sum_{m=1}^{\infty} \frac{A(m, 1)}{m^s} = \prod_p \left(1 - A(p, 1)p^{-s} + A(1, p)p^{-2s} - p^{-3s}\right)^{-1}.$$

We note that we have a bound on the Fourier coefficients.

Proposition 13.4.

$$\left| \frac{A(m_1, m_2, \ldots, m_{n-1})}{\prod_{k=1}^{n-1} m_k^{\frac{k(n-k)}{2}}} \right| = O_\alpha(1).$$

Proof. By the Fourier expansion, we have that

$$\int_0^1 \cdots \int_0^1 F(ug)\overline{\psi_{(m_1, \ldots, m_{n-1})}(u)}\, \mathrm{d}u = \frac{A(m_1, \ldots, m_{n-1})}{\prod m_k^{k(n-k)/2}} W(\ldots),$$

where

$$\psi_{(m_1, \ldots, m_{n-1})}(u) = e^{2\pi i(m_1 u_{1,2} + \cdots + m_{n-1}u_{n-1,n})}.$$

Then since the Maass form and Whittaker function are bounded and only depend on α, we are done. ☐

Hence $L(s, F)$ converges for $\mathrm{Re}(s)$ sufficiently large.
We also have the functional equation for L-function associated to a Maass form. We won't prove it, but we state it. To do so, we need to define the dual Maass form.

Definition 13.5. *Let F be a Maass form for $SL(n, \mathbb{Z})$ with Langlands parameter $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$, where $\alpha_1 + \cdots + \alpha_n = 0$. The **dual Maass form** $\widetilde{F}$ is defined to be*

$$\widetilde{F}(g) = F(w(g^{-1})^T w^{-1})$$

where w is the long element of the Weyl group of $SL(n, \mathbb{Z})$

$$w = \begin{pmatrix} & & & (-1)^{\lfloor n/2 \rfloor} \\ & & 1 & \\ & \ddots & & \\ 1 & & & \end{pmatrix}.$$

Remark 13.6. *Because F is $SL(n,\mathbb{Z})$-invariant on the left and $O(n,\mathbb{R})$-invariant on the right, we can equivalently define*

$$\widetilde{F}(g) = F((g^{-1})^T).$$

However, $w(g^{-1})^T w^{-1}$ puts the matrix $(g^{-1})^T$ into a useful Iwasawa form.

What are the Langlands parameters of $\widetilde{F}$? Considering any $D \in \mathfrak{D}^n$, we know that $D\widetilde{F}$ needs to have the same eigenvalue as $|w(g^{-1})^T w^{-1}|^{\alpha+\rho}$. We note that comparing $w(g^{-1})^T w^{-1}$ to g, the only difference is that the diagonal elements have been reversed. This corresponds to $\widetilde{F}$ having Langlands parameters $(-\alpha_n, \ldots, -\alpha_1)$.
We can now define the functional equation (for even Maass forms).

Proposition 13.7 (Functional equation)**.** *If F is an even $SL(n,\mathbb{Z})$ Maass form with Langlands parameters $(\alpha_1, \ldots, \alpha_n)$, then the completed L-function*

$$L^*(s,F) = \pi^{\frac{-ns}{2}} \prod_{j=1}^{n} \Gamma\left(\frac{s-\alpha_j}{2}\right) L(s,F)$$

satsifies the functional equation

$$L^*(s,F) = L^*(1-s,\widetilde{F}).$$

Remark 13.8. *The functional equation is much more difficult for congruent subgroups; you get ramification, and the Langlands parameters do not determine everything.*

13.3 Bump Double Dirichlet series for $\mathbf{SL}(3,\mathbb{Z})$

For a $\mathrm{SL}(3,\mathbb{Z})$ Maass form F, we can define a special double Dirichlet series first defined by Bump:

$$\sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \frac{A(m_1,m_2)}{m_1^{s_1} m_2^{s_2}}$$

Proposition 13.9.

$$\sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \frac{A(m_1,m_2)}{m_1^{s_1} m_2^{s_2}} = \frac{L(s_1,F)L(s_2,\widetilde{F})}{\zeta(s_1+s_2)}.$$

Remark 13.10. *Nothing like this exists for $GL(4)$ – this is special for $GL(3)$.*

Proof. We have

$$A(m_1,1)A(1,m_2) = \sum_{d|\gcd(m_1,m_2)} A(m_1/d, m_2/d).$$

Then

$$\begin{aligned}
L(s_1,F)L(s_2,\widetilde{F}) &= \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \frac{A(m_1,1)A(1,m_2)}{m_1^{s_1} m_2^{s_2}} \\
&= \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \frac{1}{m_1^{s_1} m_2^{s_2}} \sum_{d|\gcd(m_1,m_2)} A(m_1/d, m_2/d) \\
&= \sum_{m_1'=1}^{\infty} \sum_{m_2'=1}^{\infty} \sum_{d} \frac{1}{(m_1'd)^{s_1}(m_2'd)^{s_2}} A(m_1', m_2') \\
&= \zeta(s_1+s_2) \sum_{m_1'=1}^{\infty} \sum_{m_2'=1}^{\infty} \frac{A(m_1,m_2)}{{m_1'}^{s_1} {m_2'}^{s_2}}
\end{aligned}$$

where $m_1 = m_1'd$ and $m_2 = m_2'd$. □

Remark 13.11. *There is no good unified theory of multiple Dirichlet series with L-functions over several complex variables. Note that this isn't really a two-variable L-function - just a product of two $GL(2)$ L-functions.*

14 Lecture 14 - 3/13/25

14.1 Selberg Spectral Decomposition for SL$(2, \mathbb{Z})$

Consider $\mathcal{L}^2(\Gamma\backslash\mathfrak{h}^2)$, where $\Gamma = \mathrm{SL}(2, \mathbb{Z})$. The Selberg spectral decomposition states that

$$\mathcal{L}^2(\Gamma\backslash\mathfrak{h}^2) = \mathbb{C} \oplus \text{cusp forms} \oplus \text{Eisenstein series},$$

where the cusp forms are called the **discrete spectrum** and are countable, and the Eisenstein series are called the **continuous spectrum** and are uncountable.
We make this decomposition more explicit.

Theorem 14.1 (Selberg spectral decomposition for SL$(2, \mathbb{Z})$)**.** *Consider smooth $F \in \mathcal{L}^2(\Gamma\backslash\mathfrak{h}^2)$. Let $\eta_j(g)$ for $j = 0, 1, 3 \dots$ be an orthonormal basis of Maass forms, orthonormal with respect to the Petersson inner product, and $\eta_0(g) = \sqrt{\frac{3}{\pi}}$. Then*

$$F(g) = \sum_{j=0}^{\infty} \langle F, \eta_j \rangle \, \eta_j(g) + \frac{1}{4\pi} \int_{1/2-i\infty}^{1/2+i\infty} \langle F, E(*, s) \rangle \, E(g, s) \, \mathrm{d}s \, .$$

Remark 14.2. *Here we use the classical definition of the Eisenstein series*

$$E(g, s) = E(z, s),$$

of the form

$$E(g, s) = y^s + \phi(s) y^{1-s} + \frac{2\sqrt{y}}{\zeta^*(2s)} \sum_{n \neq 0} \sigma_{1-2s}(n) |n|^{s-1/2} K_{s-1/2}(2\pi |n| y) e^{2\pi i n x},$$

where $\zeta^*(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s) = \zeta^*(1-s)$ is the completed Riemann zeta function and

$$\phi(s) = \frac{\zeta^*(2s-1)}{\zeta^*(2s)}.$$

This will have poles (at least heuristically by RH) at Re$(s) = \frac{1}{4}$; hence we choose the contour integral to be the right of Re$(s) = \frac{1}{4}$.
We also recall that we have the functional equation

$$E(g, s) = \phi(s) E(g, 1-s)$$

and

$$\phi(s)\phi(1-s) = 1.$$

The proof will use the Mellin transform:

Definition 14.3. *Given $H : \mathbb{R} \to \mathbb{C}$ (satsifying some convergence conditions, for example for H Schwartz), the **Mellin transform** is*

$$\widetilde{H}(s) = \int_0^{\infty} H(u) u^s \frac{\mathrm{d}u}{u}$$

*and **inverse Mellin transform** is*

$$H(y) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \widetilde{H}(s) y^{-s} \, \mathrm{d}s \, ,$$

where c is chosen large enough such that $\widetilde{H}(s)$ has no poles to the right.

Proof of Selberg Spectral Decomposition. The proof will follow in two steps. Let

$$F(g) = \sum_{n \in \mathbb{Z}} A_n(y) e^{2\pi i n x}$$

be the Fourier decomposition of F, and suppose that F is orthogonal to the constant function. We will

1. Show that $\langle F, E(*, \overline{s})\rangle = \widetilde{A_0}(s-1)$.
2. Show that
$$F(g) - \frac{1}{4\pi i}\int_{1/2-i\infty}^{1/2+i\infty} \langle F, E(*, s)\rangle\, E(g,s)\,\mathrm{d}s$$
has constant term 0.

Step 1 Proof: We have that

$$\begin{aligned}
\langle F, E(*, \overline{s})\rangle &= \int_{\Gamma\backslash\mathfrak{h}^2} F\left(\begin{pmatrix} y & x \\ 0 & 1\end{pmatrix}\right)\left(\frac{1}{2}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}(\operatorname{Im}\gamma z)^s\right)\frac{\mathrm{d}x\,\mathrm{d}y}{y^2} \\
&= \frac{1}{2}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\int_{\gamma(\Gamma\backslash\mathfrak{h}^2)} F(z)(\operatorname{Im} z)^s\frac{\mathrm{d}x\,\mathrm{d}y}{y^2} \\
&= \int_0^\infty\int_0^1 F(z)y^s\frac{\mathrm{d}x\,\mathrm{d}y}{y^2} \\
&= \int_0^\infty A_0(y)y^{s-2}\,\mathrm{d}x\,\mathrm{d}y \\
&= \widetilde{A_0}(s-1),
\end{aligned}$$

where in the third line we move to an integral over the space $\Gamma_\infty\backslash\Gamma$, and the factor of $1/2$ disappears because $-I_2 \in \mathrm{SL}(2,\mathbb{Z})$ fixes $\mathfrak{h}^2$.

Step 2 Proof: Recall that the functional equation of $E(z,s)$ gives

$$E(z,s) = \phi(s)E(z,1-s).$$

Hence, we know that

$$\widetilde{A_0}(s-1) = \langle F, E(*,\overline{s})\rangle = \langle F, \phi(\widetilde{s})E(*, 1-\overline{s})\rangle = \phi(s)\widetilde{A_0}(-s),$$

and so

$$\widetilde{A_0}(-s) = \phi(1-s)\widetilde{A_0}(s-1).$$

Taking the inverse Mellin transform gives

$$A_0(y) = \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty}\widetilde{A_0}(s-1)y^{1-s}\,\mathrm{d}y$$

and applying the above identity gives

$$A_0(y) = \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty}\widetilde{A_0}(-s)\phi(s)y^{1-s}\,\mathrm{d}y\,.$$

We also can apply the transformation $1-s \to s$ to the original inverse Mellin transform to get

$$A_0(y) = \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty}\widetilde{A_0}(-s)y^{s}\,\mathrm{d}y$$

Adding the two equations together gives

$$A_0(y) = \frac{1}{4\pi i}\int_{1/2-i\infty}^{1/2+i\infty}\widetilde{A_0}(-s)(y^s + \phi(s)y^{1-s})\,\mathrm{d}y\,.$$

For $\operatorname{Re}(s) = 1/2$, note that $\widetilde{A_0}(-s) = \widetilde{A_0}(\overline{s}-1)$, and by the first step we know that

$$\widetilde{A_0}(-s) = \widetilde{A_0}(\overline{s}-1) = \langle F, E(*,s)\rangle\,.$$

Thus we get that

$$A_0(y) = \frac{1}{4\pi i} \int_{1/2-i\infty}^{1/2+i\infty} \langle F, E(*, s) \rangle \left(y^s + \phi(s) y^{1-s} \right) \mathrm{d}y \,,$$

where we note that $y^s + \phi(s)y^{1-s}$ is precisely the constant term of $E(g, s)$. Since the other parts of the expansion will contribute to the nonconstant terms in the Fourier expansion, we conclude that

$$F(g) - \frac{1}{4\pi i} \int_{1/2-i\infty}^{1/2+i\infty} \langle F, E(*, s) \rangle E(g, s) \, \mathrm{d}s$$

has constant term 0, as desired. □

14.2 Selberg Spectral Decomposition for SL$(n, \mathbb{Z})$

How do we generalize the decomposition to SL$(n, \mathbb{Z})$?
Recall that we have the parabolic subgroups

$$\mathcal{P}_{n_1,\ldots,n_r} = \left\{ \begin{pmatrix} \mathrm{GL}(n_1) & & & * \\ & \mathrm{GL}(n_2) & & \\ & & \ddots & \\ & & & \mathrm{GL}(n_r) \end{pmatrix} \right\}$$

corresponding to partitions of $n = n_1 + \cdots + n_r$. We also defined the power function

$$|g|^s = \prod_{i=1}^{r} |\det(m_i)|^{s_i}$$

for $g \in \begin{pmatrix} m_1 & & & * \\ & m_2 & & \\ & & \ddots & \\ & & & m_r \end{pmatrix} \in \mathcal{P}_{n_1,\ldots,n_r}$, where $s = (s_1, \ldots, s_r) \in \mathbb{C}^r$, with $\sum n_i s_i = 0$. We also defined an induced Maass form

$$\Phi(g) = \prod_{i=1}^{r} \phi_i(m_i),$$

where each $\phi_i \in \mathcal{L}^2(\Gamma_n \backslash \mathfrak{h}^n)$.
With these, we defined the Langlands Eisenstein series

$$E_{\mathcal{P}_{n_1,\ldots,n_r},\Phi}(g, s) = \sum_{\gamma \in (\Gamma_n \cap \mathcal{P}_{n_1,\ldots,n_r}) \backslash \Gamma_n} \Phi(\gamma g) |\gamma g|^{s+\rho}.$$

Langlands and Arthur showed a corresponding spectral decomposition for $\mathcal{L}^2(\mathrm{SL}(n, \mathbb{Z}) \backslash \mathfrak{h}^n)$:

Theorem 14.4 (Langlands-Arthur)**.** *Let $\phi_1, \phi_2, \ldots$ be an orthonormal basis for cusp forms for SL$(n, \mathbb{Z})$. Let $F \in \mathcal{L}^2(SL(n, \mathbb{Z}) \backslash \mathfrak{h}^n)$ be such that F is orthogonal to the residual spectrum (residues of Eisenstein series in the s variable). Then*

$$F(g) = \sum_{j=1}^{\infty} \langle F, \phi_j \rangle \phi_j(g) + \sum_{\mathcal{P} = \mathcal{P}_{n_1,\ldots,n_r}} \sum_{\Phi} \iint_{\substack{n_1 s_1 + \cdots + n_r s_r = 0 \\ Re(s_j) = 0}} \langle F, E_{\mathcal{P},\Phi}(*, s) \rangle E_{\mathcal{P},\Phi}(g, s) \, \mathrm{d}s \,,$$

where $\mathrm{d}s = \prod_{i=1}^{r} \mathrm{d}s_r$.

Remark 14.5. *In SL$(2, \mathbb{Z})$, $E(z, s) = y^s + \phi(s) y^{1-s} + \ldots$ has a pole at $s = 1$, with constant residue (as $y^0 = 1$). Hence the residual spectrum in the SL$(2, \mathbb{Z})$ case are precisely the constant functions.*

Example 14.6. *For SL$(4, \mathbb{Z})$, let $\Phi = (\phi_1, \phi_1)$, two GL(2) cuspforms. We will also have a function $\phi(s)$ with is a ratio of $L(s, \phi_1 \times \overline{\phi_1})$. This will give the first interesting example of a residual spectrum.*

14.3 Kuznetsov Trace Formula

For the next week, we will talk about the Kuznetsov Trace formula. We want to construct an automorphic form that is not an Eisenstein series, and apply the spectral expansion to it. In particular, the automorphic forms we wish to define are the Poincare series.

Definition 14.7 (Poincare Series)**.** *For any $M = (m_1, \ldots, m_{n-1}) \in \mathbb{Z}^{n-1}$, we have the* ***Poincare series***

$$P^M(g,s) = \sum_{\gamma \in U_n \backslash \Gamma_n} |\gamma g|^{s+\rho} \Psi_M(\gamma g),$$

where $\Gamma_n = SL(n,\mathbb{Z})$, U_n is the group of unipotent matrices, and

$$\Psi_M(g) = \Psi_M(x) = e^{2\pi i (m_1 x_{1,2} + m_2 x_{2,3} + \cdots + m_{n-1} x_{n-1,n})},$$

where $g = xy$ is the Iwasawa decomposition.

Remark 14.8. *If $M = (0, \ldots, 0)$, we recover the Eisenstein series.*

Remark 14.9. *This can be generalized to any parabolic subgroup.*

Note that for any $\mathrm{SL}(n,\mathbb{Z})$ Maass form ϕ and any $M = (m_1, \ldots, m_{n-1})$,

$$\begin{aligned}\left\langle \phi, P^M(g,\overline{s}) \right\rangle &= \int_{\Gamma_n \backslash \mathfrak{h}^n} \phi(g) \sum_{\gamma \in U_n \backslash \Gamma_n} |\gamma g|^{s+\rho} \Psi_{(m,1,\ldots,1)}(\gamma g) \mathrm{d}^* g \\ &= \int_{y_1=0}^{\infty} \cdots \int_{y_n=0}^{\infty} \int_0^1 \cdots \int_0^1 \phi(g) \psi_{(m,1,\ldots,1)}(x) |y|^{s+\rho} \mathrm{d}^* g,\end{aligned}$$

where $\mathrm{d}^* g$ is the $\mathrm{GL}(n,\mathbb{R})$-invariant measure. Note that if $M = (m, 1, \ldots, 1)$ (and ϕ a Hecke-Maass form), this integral is precisely $A(m, 1, \ldots, 1)$ times the Mellin transform of a Whittaker function – note that this is the same as what happens with $M = (0, \ldots, 0)$ (i.e. when the Poincare series is the Eisenstein series).

Remark 14.10. *The Mellin transform of a Whittaker function is understood for $SL(2,\mathbb{Z})$ and $SL(3,\mathbb{Z})$, but not well understood for $n \geq 4$. In these cases, we can include a test function in $P^M(g,\overline{s})$ to get better information about the integral of the Whittaker function.*

One application of the Kuznetsov trace formula is to get information about the average value of the jth coefficient of a Maass form. We will talk about this application in the future.

Remark 14.11. *In the spectral expansion generally, the most difficult terms to compute are the continuous spectrum, which will typically affect the error terms. Next time, we'll talk about some applications, which give better results than considering this from the representation theoretic perspective, because we tackle the continuous spectrum directly.*

15 Lecture 15 - 3/25/25

15.1 Functional Equation of the Maximal Parabolic Langlands Eisenstein Series

Recall that we have the parabolic subgroup

$$P_{n_1,\dots,n_r} = \left\{ \begin{pmatrix} \mathrm{GL}(n_1) & & & * \\ & \mathrm{GL}(n_2) & & \\ & & \ddots & \\ & & & \mathrm{GL}(n_r) \end{pmatrix} \right\}$$

with Langlands decomposition

$$g = \begin{pmatrix} I_{n_1} & & & * \\ & I_{n_2} & & \\ & & \ddots & \\ & & & I_{n_r} \end{pmatrix} \begin{pmatrix} m_1 & & & \\ & m_2 & & \\ & & \ddots & \\ & & & m_n \end{pmatrix}.$$

For $s = (s_1, \dots, s_r)$ with $\sum n_i s_i = 0$, we have the power function

$$|g|^s_{P_{n_1,\dots,n_r}} = \prod_{i=1}^r |\det(\mathfrak{m}_i(g))|^{s_i}$$

and Langlands Eisenstein series

$$E_{P_{n_1,\dots,n_r}}(g,s) = \sum_{\gamma \in (P_{n_1,\dots,n_r} \cap \mathrm{SL}(n,\mathbb{Z})) \backslash \mathrm{SL}(n,\mathbb{Z})} |\gamma g|^{s+\rho},$$

where $\rho_i = \frac{n-n_i}{2} - n_1 - \cdots - n_{i-1}$.
Our goal is to completely understand the Eisenstein series; i.e. we want to get their Fourier coefficients and the functional equation.
There is one case where we can get everything via Poisson summation - the case of the maximal parabolic subgroup of $\mathrm{SL}(n,\mathbb{Z})$ (the maximal compact subgroup, also known as the mirabolic subgroup):

$$P_{n-1,1} = \begin{pmatrix} \mathrm{GL}(n-1) & * \\ & 1 \end{pmatrix}.$$

In this case, we have $s = s_1 + s_2$ with $(n-1)s_1 + s_2 = 0$, with $\rho = (\frac{1}{2}, -\frac{n-1}{2})$. Hence

$$E_{P_{n-1,1}}(g,s) = \sum_{\gamma \in (P_{n-1,1} \cap \Gamma_n) \backslash \Gamma_n} (\det \gamma g)^{s_1 + 1/2}.$$

Theorem 15.1. *$E_{P_{n-1,1}}(g,s)$ has meromorphic continuation to all of $s_i \in \mathbb{C}$, with simple poles at $s_1 = \pm 1/2$. In particular, we also have the completed Eisenstein series*

$$E^*_{P_{n-1,1}}(g,s) = \pi^{-\frac{n(s_1+1/2)}{2}} \Gamma\left(\frac{n(s_1+1/2)}{2}\right) \zeta(n(s_1+1/2)) E_{P_{n-1,1}}(g,s)$$

with functional equation

$$E^*_{P_{n-1,1}}(g,(s_1,s_2)) = E^*_{P_{n-1,1}}((g^{-1})^T, (-s_1,-s_2)).$$

Proof. We can prove this functional equation using Poisson summation: for a smooth function $f : \mathbb{R}^n \to \mathbb{C}$ and $g \in \mathrm{GL}(n,\mathbb{R})$,

$$\sum_{m \in \mathbb{Z}^n} f(mg) = \frac{1}{|\det g|} \sum_{m \in \mathbb{Z}^n} \widehat{f}(m(g^{-1})^T),$$

where

$$\widehat{f}((x_1,\ldots,x_n)) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f((t_1,\ldots,t_n))e^{-2\pi i(t_1x_2+\cdots+t_nx_n)}\,\mathrm{d}t_1\ldots\mathrm{d}t_n\,.$$

For $\gamma,\gamma'\in(P_{n-1,1}\cap\Gamma_n)\backslash\Gamma_n$, we note that for $p\gamma=\gamma'$ with $p\in P_{n-1,1}$, this occurs iff the last rows of γ' and γ share the same greatest common factor. Thus we can write

$$\gamma g = \begin{pmatrix} & * & \\ a_1 & \ldots & a_n\end{pmatrix}\begin{pmatrix}1 & & x_{ij}\\ & \ddots & \\ & & 1\end{pmatrix}\begin{pmatrix}y_1\ldots y_{n-1} & & & \\ & \ddots & & \\ & & y_1 & \\ & & & 1\end{pmatrix} = \begin{pmatrix} & * & \\ b_1 & \ldots & b_n\end{pmatrix}$$

What do the elements in the last row look like? We can compute that

$$\begin{aligned} b_1 &= a_1y_1\ldots y_{n-1}\\ b_2 &= (a_1x_{12}+a_2)y_1\ldots y_{n-1}\\ &\vdots\\ b_n &= a_1x_{1,n}+a_2x_{2,n}+\cdots+a_{n-1}x_{n-1,n}+a_n\end{aligned}$$

We can now rewrite the Eisenstein series in terms of these new coefficients, which will give an Epstein zeta function.

However, this previous formulation for $\gamma\cdot g$ is not in Iwasawa form, and must be converted to get the formula for the Eisenstein series. Let $\gamma\cdot g=\tau k(rI_n)$, where τ is the Iwasawa form for $\gamma\cdot g$ (i.e. $k\in O(n,\mathbb{R})$ and $r\in\mathbb{R}$). We want to compute

$$\det(\gamma\cdot g)^s = (\det\tau)^s.$$

Note that $\gamma g(\gamma g)^T = r^2\tau\tau^T$. This implies that

$$b_1^2+\cdots+b_n^2 = r^2$$

by examining the bottom right element on both sides.

Thus

$$\det(\gamma\cdot g) = \det\tau = \frac{\det\gamma}{\det g}r^{-n} = \det(g)(b_1^2+\cdots+b_n^2)^{-n/2},$$

so we can write

$$E_{P_{n-1,1}}(g,s) = (\det g)^{s_1+1/2}\sum_{\substack{(a_1,\ldots,a_n)\neq 0\\ (a_1,\ldots,a_n)=1}}(b_1^2+\cdots+b_n^2)^{-\frac{n(s_1+1/2)}{2}}.$$

Multiplying by $\zeta\left(\frac{n(s_1+1/2)}{2}\right)$ removes the relatively prime condition. Hence

$$\zeta\left(\frac{n(s_1+1/2)}{2}\right)E_{P_{n-1,1}}(g,s) = \sum_{(a_1,\ldots,a_n)\neq 0}(b_1^2+\cdots+b_n^2)^{-\frac{n(s_1+1/2)}{2}}.$$

Now we apply Poisson summation to get the meromorphic continuation. Let

$$f_u((x_1,\ldots,x_n)) = e^{-\pi(x_1^2+\cdots+x_n^2)u}.$$

Then

$$E^*_{P_{n-1,1}}(g,s) = |\det(g)|^{s_1+1/2}\int_0^\infty\left(\sum_{(a_1,\ldots,a_n)\in\mathbb{Z}^n} f_u((a_1,\ldots,a_n)g) - f_u((0,\ldots,0))\right)u^{\frac{n(s_1+1/2)}{2}}\frac{\mathrm{d}u}{u},$$

where one can now finish as in the functional equation of the Riemann zeta function (by splitting the integral into the integral from 0 to 1 and the integral from 1 to infinity). The poles arise from the $f_u((0,\ldots,0))$ term. □

Remark 15.2. *The Langlands functional equation is*

$$E^*_{P_{n-1,1}}(g,(s_1,s_2)) = E^*_{P_{1,n-1}}(g,(s_2,s_1)).$$

It turns out be equivalent to the functional equation we have above.

15.2 Rankin-Selberg for $\mathbf{GL}(n) \times \mathbf{GL}(n)$

Let ϕ_1, ϕ_2 be (even, for simplicity) Maass forms for $\mathrm{SL}(n, \mathbb{Z})$, with Hecke coefficients $A_{\phi_1}(n)$ and $A_{\phi_2}(n)$, and corresponding L-functions

$$L(w, \phi_1) = \sum_n \frac{A_{\phi_1}(n)}{n^w}$$

and

$$L(w, \phi_2) = \sum_n \frac{A_{\phi_2}(n)}{n^w}.$$

Definition 15.3. *The **Rankin-Selberg L-function** is defined to be*

$$L(w, \phi_1 \times \phi_2) = \sum_{m_1=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} \frac{A_{\phi_1}(m_1, \ldots, m_{n-1})\overline{A_{\phi_2}(m_1, \ldots, m_{n-1})}}{(m_1^{n-1} m_2^{n-2} \ldots m_{n-1})^w}$$

which converges absolutely for $Re(w)$ sufficiently large.

One can can use the Euler products of the Maass forms to get the Euler product of the Rankin-Selberg L-function.

Proposition 15.4. *If ϕ_1 and ϕ_2 have Euler products*

$$L(w, \phi_1) = \prod_p \prod_{i=1}^{n} (1 - \alpha_{p,i} p^{-w})^{-1}$$

and

$$L(w, \phi_2) = \prod_p \prod_{i=1}^{n} (1 - \beta_{p,i} p^{-w})^{-1},$$

respectively, then the Euler product for the Rankin-Selberg L-function is

$$L(w, \phi_1 \times \phi_2) = \prod_p \prod_{i=1}^{n} \prod_{j=1}^{n} (1 - \alpha_{p,i} \overline{\beta_{p,j}} p^{-w})^{-1}$$

Proof. See Dorian's book – don't think it was covered in class. ☐

Remark 15.5. *When you work adelically, you don't see the L-function coefficients - you only see the Euler product. Hence if you want to work with Euler products, its better to work adelically.*

Dorian never stated this is what we trying to prove in class, but we have the functional equation of the Rankin-Selberg L-function.

Proposition 15.6. *If ϕ_1, ϕ_2 are of Langlands parameters $(\alpha_1, \ldots, \alpha_n)$ and $(\beta_1, \ldots, \beta_n)$, respectively, with completed L functions*

$$L^*(w, \phi_1) = \pi^{-\frac{nw}{2}} \prod_{i=1}^{n} \Gamma\left(\frac{w - \alpha_i}{2}\right) L(w, \phi_1)$$

and

$$L^*(w, \phi_2) = \pi^{-\frac{nw}{2}} \prod_{i=1}^{n} \Gamma\left(\frac{w - \beta_i}{2}\right) L(w, \phi_2),$$

then $L(w, \phi_1 \times \phi_2)$ has meromorphic continuation to all $w \in \mathbb{C}$, with at most a simple pole at $s = 1$ with residue proportional to $\langle \phi_1, \phi_2 \rangle$, and we have the completed Rankin-Selberg L-function

$$L^*(w, \phi_1 \times \phi_2) = \pi^{-\frac{n^2 s}{2}} \prod_{i=1}^{n} \prod_{j=1}^{n} \Gamma\left(\frac{s - \alpha_i - \overline{\beta_j}}{2}\right) L(w, \phi_1 \times \phi_2)$$

with functional equation

$$L^*(w, \phi_1 \times \phi_2) = L^*(1 - w, \overline{\phi_1} \times \overline{\phi_2}).$$

Proof. To compute the functional equation of the L-function, like the classical GL(2) case, we will want to use the inner product

$$\left\langle \phi_1, \phi_2 E_{P_{n-1},1}(g, \overline{s}) \right\rangle = \int_{\Gamma_n \backslash \mathfrak{h}^n} \phi_1(g) \overline{\phi_2(g) E_{P_{n-1},1}(g, \overline{s})} \, \mathrm{d}g \, .$$

Remark 15.7. *Note that this matches exactly what we expect for GL(2) (the Eisenstein series is just the standard GL(2) Eisenstein series).*

We need to apply unfolding, but here we will need to unfold twice (once for the Eisenstein series, and once for the Fourier expansion of GL(n) Maass forms). First, unfolding the Eisenstein series definition gives

$$\int_{P_{n-1,1} \backslash \mathfrak{h}^n} \phi_1(g) \overline{\phi_2(g)} |g|_{P_{n-1,1}}^{s_1+1/2} \, \mathrm{d}g \, .$$

Now, we can write

$$g = \begin{pmatrix} g' & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & & & r_1 \\ & \ddots & & \vdots \\ & & 1 & r_{n-1} \\ & & & 1 \end{pmatrix} .$$

Thus we can take the union

$$\bigcup_{\gamma \in U_{n-1}(\mathbb{Z}) \backslash \mathrm{SL}(n-1, \mathbb{Z})} \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} (P_{n-1,1} \backslash \mathfrak{h}^n) \cong U_n(\mathbb{Z}) \backslash \mathfrak{h}^n .$$

The left union appears in the Fourier expansion of Maass forms. Working this out gives

$$\left\langle \phi_1, \phi_2 E_{P_{n-1,1}}(g, \overline{s}) \right\rangle = \sum_{(m_1, \ldots, m_{n-1})} \int_{U_n(\mathbb{Z}) \backslash \mathfrak{h}^n} \phi_1(g) \frac{A_{\phi_2}(m_1, \ldots, m_{n-1})}{*} W(g) |g|^{s_1+1/2} \, \mathrm{d}g \, .$$

The Whittaker function contains an exponential function, and hence picks off the corresponding coefficient in ϕ_1. This gives

$$\sum_{m_1=1}^{\infty} \cdots \sum_{m_{n-1}=1}^{\infty} \frac{A_{\phi_1}(m_1, \ldots, m_{n-1}) \overline{A_{\phi_2}(m_1, \ldots, m_{n-1})}}{\prod_{k=1}^{n-1} m_k^{k(n-k)}} \int_0^\infty \cdots \int_0^\infty W_\alpha(My) \overline{W_\beta}(My) (\det y)^{s_1+1/2} \mathrm{d}^* y,$$

where α and β are the Langlands parameters for ϕ_1 and ϕ_2, respectively, and $M = \begin{pmatrix} m_1 \ldots m_{n-1} & & & \\ & \ddots & & \\ & & m_1 & \\ & & & 1 \end{pmatrix}$.

By a change of a variables, this becomes a product of the Rankin-Selberg L-function and the Mellin transform of a product of Whittaker functions, which by a result of Eric Stade is a product of Gamma factors, and this result, along with the meromorphic continuation/functional equation of $E_{P_{n-1,1}}(g, s)$, gives the functional equation as desired. For more details, see Dorian's book (section 12.1). $\square$

16 Lecture 16 - 3/27/25

16.1 Normalized Whittaker Function

Today we write down the best definition of a Whittaker function.

Definition 16.1. *Let $\alpha = (\alpha_1, \dots, \alpha_n)$ be a Langlands parameter. Then we define the* ***normalized Whittaker function***

$$W_\alpha(g) = \prod_{1\le j<k\le n} \frac{\Gamma\left(\frac{1+\alpha_j-\alpha_k}{2}\right)}{\pi^{\frac{1+\alpha_j-\alpha_k}{2}}} \int_{U_n(\mathbb{R})} |w_n ug|_B^{\alpha+\rho_B} \overline{\psi_{1,\dots,1}}(u)\, \mathrm{d}u\,,$$

where $\mathrm{d}u = \prod_{1\le j<k\le n} \mathrm{d}u_{ij}$*,* B_n *is the Borel subgroup, and*

$$w_n = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{pmatrix}$$

be the long element of the Weyl group.

Remark 16.2. *We can replace the Jacquet-Whittaker functions in the previous Fourier expansion for the Maass forms with these normalize ones; hence the Fourier coefficients will differ by the product of the Γ factors and the πs. This doesn't affect any theorems, as this product is a constant independent of which coefficient is chosen. This choice, however, will affect the value of the "first coefficient" because of different normalizations.*
We will make this change from now on; i.e. all our Fourier coefficients will divided by the product of these Gammas and multiplied by the πs from what we had before.

Let $\sigma \in S_n$ be a permutation, and let $\sigma(\alpha) = (\alpha_{\sigma(i)})_{i=1}^n$.

Proposition 16.3. *The functional equation of the Whittaker function is*

$$W_{\sigma(\alpha)}(g) = W_\alpha(g).$$

Sketch. The power function only depends on the diagonal elements. A permutation of the α hence corresponds to permuting the diagonal elements. One can also express this as a conjugation by an element of the Weyl group; it suffices to prove the functional equation for permutations that correspond to a swap of two adjacent elements, which will reduce to essentially the GL(2) case. For more details, the proof can be found in Dorian's book (although written in spectral rather than Langlands function). ☐

Remark 16.4. *The generic representations are very important, and by definition have Whittaker functions (at the archimedean place, which corresponds). The choice of the Γ factors are necessary to show that the Whittaker function never vanishes for all choices of Langlands parameters, which is needed for the generic representations.*

16.2 First Coefficient of Langlands Eisenstein series

Our goal is to compute the first coefficient for every Langlands Eisenstein series. To do so, we'll need to compute the first coefficient for any Maass forms, as Langlands Eisenstein series are induced by Maass forms. Last time, we talked about the Mellin transform of the product of two Whittaker functions.

Theorem 16.5 (Stade)**.** *For $s \in \mathbb{C}$,*

$$\int_0^\infty \cdots \int_0^\infty W_\alpha(y)\overline{W_\beta}(y) \det(y)^s \,\mathrm{d}y = \frac{\prod_{j=1}^n \prod_{k=1}^n \Gamma\left(\frac{s+\alpha_j-\beta_k}{2}\right)}{2\pi^{\frac{sn(n-1)}{2}}\Gamma\left(\frac{ns}{2}\right)}$$

where y is the usual diagonal matrix $\begin{pmatrix} y_1\cdots y_{n-1} & & & \\ & \ddots & & \\ & & y_1 & \\ & & & 1 \end{pmatrix}$ *and* $\mathrm{d}y = \prod_{k=1}^{n-1} \frac{\mathrm{d}y_k}{y_k^{k(n-k)+1}}$. *Here note that* $(\det y)^s = \prod_{j=1}^{n-1} y_j^{(n-j)s}$.

Recall last time, we considered $s = (s_1, s_2)$ with $(n-1)s_1 + s_2 = 0$, and we showed that

$$\begin{aligned}
&\zeta(n(s_1+1/2)) \left\langle \phi_1, \phi_2 E_{P_{n-1,1}}(*, \overline{s}) \right\rangle \\
&= \sum_{m_1=1}^{\infty} \cdots \sum_{m_{n-1}=1}^{\infty} \frac{A_{\phi_1}(m_1,\ldots,m_{n-1})\overline{A_{\phi_2}(m_1,\ldots,m_{n-1})}}{(m_1^{n-1}m_2^{n-2}\ldots m_{n-1})^{s_1+1/2}} \left\langle W_{\alpha_{\phi_1}}, W_{\alpha_{\phi_2}} \det(\cdot)^{s_1+1/2} \right\rangle \\
&= A_{\phi_1}(1,\ldots,1)\overline{A_{\phi_2}(1,\ldots,1)} L(s_1+1/2, \phi_1 \times \overline{\phi_2}) \left\langle W_{\alpha_{\phi_1}}, W_{\alpha_{\phi_2}} \det(\cdot)^{s_1+1/2} \right\rangle,
\end{aligned}$$

where the second inner product is given to us by the result of Stade. Note that implicitely we assume normalized coefficients to get the L-function, which explains the $A_{\phi_1}(1,\ldots,1)\overline{A_{\phi_2}(1,\ldots,1)}$ term. We can now use this to get information about the first coefficient of a Maass form:

Theorem 16.6. *Let ϕ be a Maass form for $SL(n,\mathbb{Z})$ with Langlands parameter $\alpha = (\alpha_1,\ldots,\alpha_n)$. Then for some constant c_n depending only on n,*

$$|A_\phi(1,\ldots,1)|^2 = \frac{c_n \langle \phi, \phi \rangle}{L^*(1, Ad\ \phi)},$$

where

$$L(w, Ad\ \phi) = \frac{L(w, \phi \times \overline{\phi})}{\zeta(w)}$$

and

$$L^*(w, Ad\ \phi) = \left(\prod_{1 \le j \ne k \le n} \Gamma\left(\frac{1+\alpha_j - \alpha_k}{2} \right) \right) L(1, Ad\ \phi)$$

Proof. $E_{P_{n-1},1}(g,s)$ has poles at $s = \pm 1/2$. Suppose $R = \mathrm{Res}_{s=1/2} E_{P_{n-1,1}}(g,s)$ (one can show that this residue does not depend on g). Then

$$\langle \phi, \phi \rangle = \int_{\mathrm{SL}(n,\mathbb{Z})\backslash \mathfrak{h}^n} |\phi(g)|^2 \frac{\mathrm{Res}_{s=1/2} E(g,s)}{R} \, \mathrm{d}g\,.$$

On the other hand, using the inner product computation from before to the RHS and taking the residue at $s_1 = 1/2$ gives

$$\frac{|A_\phi(1,\ldots,1)|^2}{R} \mathrm{Res}_{s_1=1/2} \left(\frac{L(s_1+1/2, \phi \times \overline{\phi})}{\zeta(n(s_1+1/2))} \langle \mathrm{STADE} \rangle \right).$$

Applying Stade's formula finishes the proof. ☐

Remark 16.7. *If we normalize ϕ such that $\langle \phi, \phi \rangle = 1$, we get a simpler formula – we'll make this assumption later.*

Remark 16.8. *The functional equation adelically only depends on the Langlands parameters at ∞. Then the minimal parabolic Eisenstein series can be used to determine the functional equation for any Maass form – the Gamma factors are exactly the same. The only thing that changes is the pth coefficient. This can be done on any Chevellay group. Dorian calls this the template method – for more details, see this paper of Goldfeld, Miller, and Woodbury.*

We will use the template method to get the first coefficient of every parabolic Eisenstein series. We'll use the Borel Eisenstein series as the template. One can do this instead with the Bruhat decomposition and Kloosterman sums, but that requires a tedious computation.

16.3 First Coefficient of Borel Eisenstein Series

Let $s = (s_1, \ldots, s_n)$, and recall that we have the Borel (minimal parabolic) Eisenstein series

$$E_{B_n}(g, s) = \sum_{\gamma \in (B_n \cap \Gamma_n) \backslash \Gamma_n} |\gamma g|^{s + \rho_{B_n}}.$$

One can show that we have the first coefficient (due to Selberg)

$$A_{E_{B_n}}((1, \ldots, 1), s) = c_n \prod_{1 \le j < k \le n} \zeta^*(1 + s_j - s_k)^{-1}.$$

A (adelic) proof can be found in this paper of Goldfeld, Miller, and Woodbury. This formula will be the template for other first coefficients of Eisenstein series. To do so, however, we need to know the Langlands parameters of the most general Langlands Eisenstein series.
Let $P_{n_1,\ldots,n_r}$ be a parabolic subgroup, $(s_1, \ldots, s_r) = s \in \mathbb{C}^r$, and $\Phi = \phi_1 \otimes \cdots \otimes \phi_r$, where each ϕ_i is an $\mathrm{SL}(n_i, \mathbb{Z})$ Maass form. Then

$$E_{P_{n_1,\ldots,n_r},\Phi}(g, s) = \sum_{\gamma \in (P_{n_1,\ldots,n_r} \cap \Gamma_n) \backslash \Gamma_n} \Phi(\gamma g) |\gamma g|^{s+\rho}_{\mathcal{P}_{n_1,\ldots,n_r}}.$$

Assume that $\langle \phi_i, \phi_i \rangle = 1$ for all i, and suppose each ϕ_j has Langlands parameter $\alpha^{(j)} = (\alpha_{j,1}, \ldots, \alpha_{j,n_j})$.

Proposition 16.9. *The Langlands parameters $\alpha_{P,\Phi}(s)$ of $E_{P_{n_1,\ldots,n_r},\Phi}$ are*

$$(\alpha_{1,1} + s_1, \ldots, \alpha_{1,n_1} + s_1, \alpha_{2,1} + s_2, \ldots, \alpha_{2,n_2} + s_2, \ldots, \alpha_{r,1} + s_r, \ldots, \alpha_{r,n_r} + s_r)$$

Proof. We need to prove that

$$E_{P_{n_1,\ldots,n_r},\Phi}(*, s)$$

has the same eigenvalue of all $\mathrm{GL}(n, \mathbb{R})$ invariant differential operators as

$$|*|_{B_n}^{\alpha_{P,\Phi}(s) + \rho_{B_n}}.$$

One can in fact show that

$$|*|_{P_{n_1,\ldots,n_r}}^{s + \rho_{P_{n_1,\ldots,n_r}}} = |*|_{B_n}^{\alpha_{P,\Phi}(s) + \rho_{B_n}}$$

by checking diagonal elements.
Taking care of the Maass forms requires a brute force computation.

□

Remark 16.10. *How does the template method work? Assume that we have an induced Maass form Φ. The first thing to do with is to replace Φ with a Borel Eisenstein series with the same Langlands parameter. Once this is done, the new object will be a minimal parabolic Eisenstein series for some choice of Langlands parameters. Computing these will allow us to compute its first coefficient, up to some normalization factor. What is this factor? Since we normalized by $\langle \phi_i, \phi_i \rangle = 1$, we will need to multiply by some factor corresponding to the adjoint L-function. Hence the first coefficient will correspond to a product of completed ζ functions and adjoint L-functions.*

Next time, we will perform this technique in detail.

17 Lecture 17 - 4/1/25

17.1 Template Method for First Fourier Coefficients of Eisenstein Series

Last time we discussed the template method, which serves as a simple algorithm for computing the first coefficient of Langlands Eisenstein series. We describe it in more detail today.
Recall that given a parabolic subgroup $P_{n_1,\dots,n_r}$, $s = (s_1, \dots, s_r) \in \mathbb{C}^r$ with $\sum s_i r_i = 0$, and $\Phi = \phi_1 \otimes \cdots \otimes \phi_r$, with each ϕ_j an $\mathrm{SL}(n_j, \mathbb{Z})$ Maass form, we have the Langlands Eisenstein series

$$E_{P,\Phi}(g, s) = \sum_{(\Gamma_n \cap P)\backslash \Gamma_n} \Phi(\gamma g) |\gamma g|_P^{s+\rho}.$$

Take $M = (m_1, \dots, m_{n-1}) \in \mathbb{Z}^{n-1}$. We have the character $\psi_m(u)$ on $u = \begin{pmatrix} 1 & & & u_{ij} \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \in U_n(\mathbb{R})$ of the form

$$\psi_M(u) = e^{2\pi i (m_1 u_{1,2} + \cdots + m_{n-1} u_{n-1,n})}.$$

We can extract the mth coefficient via the integral

$$\int_0^1 \cdots \int_0^1 E_{P,\Phi}(ug, s) \overline{\psi_M(u)} \, \mathrm{d}u = A_{P,\Phi}(M, s) \cdot \text{Whittaker function}.$$

Here the $A_{P,\Phi}(M, s)$ is the mth coefficient. It splits into the form

$$A_{P,\Phi}(M, s) = A_{P,\Phi}((1, \dots, 1), s) \lambda_{P,\Phi}(M, s),$$

where $\lambda_{P,\Phi}(M, s)$ is the Hecke eigenvalue, and $A_{P,\Phi}((1, \dots, 1), s)$ is the first coefficient.

Remark 17.1. *We only have a nice formula for the eigenvalue of Hecke operator in the case* $M = (m, 1, \dots, 1)$, *and the multiplicativity relations give all the other eigenvalues.*

Here is the formula for the first coefficient of a Langlands Eisenstein series:

Theorem 17.2. *Assume the* ϕ_i *are normalized such that* $\langle \phi_i, \phi_i \rangle = 1$. *Then*

$$A_{P,\Phi}((1, \dots, 1), s) = c_n \left(\prod_{j=1}^{r} L^*(1, Ad\ \phi_j)^{-\frac{1}{2}} \right) \left(\prod_{1 \le j < k \le n} L^* \left(1 + s_j - s_k, \phi_j \times \phi_k\right)^{-1} \right),$$

for some constant c_n *depending only on* n, *and if one of the* n_j *or* n_k *is 1, then the Rankin-Selberg* L*-function is replaced by the completed* L *function for a Maass form, and if* $n_j = n_k = 1$, *the Rankin-Selberg* L*-function is replaced the completed Riemann zeta function* $\zeta^*(1 + s_j - s_k)$.

Recall that the adjoint L-function was defined to be

$$L(s, \mathrm{Ad}\ \phi) = \frac{L(s, \phi \times \overline{\phi})}{\zeta(s)}$$

with completed adjoint L-function

$$L^*(1, \mathrm{Ad}\ \phi) = \left(\prod_{1 \le j \ne k \le r} \Gamma\left(\frac{1 + \alpha_j - \alpha_k}{2} \right) \right) L(1, \mathrm{Ad}\ \phi)$$

Remark 17.3. *The adjoint* L*-function at* 1 *is essentially the residue of the Rankin-Selberg Eisenstein series at* 1.

Remark 17.4. *Next week, we will talk about the Kuznetsov trace formula, which has a contribution from the continuous spectrum - i.e. we will sum over integrals against the Eisenstein series. We want a power-saving error term, so we'll need to understand the size of the Eisenstein series coefficients - this is the motivation for why we care about computing the first coefficient of the Eisenstein series.*
We have good estimates for the Rankin-Selberg L-functions, but not for the adjoint L-functions. Conjecturally, we believe that $c_j^{-\varepsilon} \ll |L(1, Ad\ \phi_j)| \ll c_j^{\varepsilon}$*, where* c_j *is the conductor of* ϕ_j*, which is essentially the sum of the squares of the Langlands parameters/the Laplace eigenvalue. The conjecture (lower bound, the hard part) is proved for GL(2) due to Iwaniec – the lower bound is essentially equivalent to proving an analogue of the prime number theory for the adjoint L-function.*

Remark 17.5. *The proof of this template method uses the adelic perspective, and can be found in this paper of Goldfeld, Miller, and Woodbury.*

Let B_n be the Borel Eisenstein series; in this case, we know the first coefficient is due to Selberg (up to a constant)

$$A_{B_n}((1,\ldots,1),s) = \prod_{1\le j<k\le n} \zeta^*(1+s_j-s_k).$$

Recall that the Langlands parameters of the general Eisenstein series were of the form

$$(\alpha_{1,1}+s_1,\ldots,\alpha_{1,n_1}+s_1,\alpha_{2,1}+s_2,\ldots,\alpha_{2,n_2}+s_2,\ldots,\alpha_{r,1}+s_r,\ldots,\alpha_{r,n_r}+s_r),$$

where $\alpha_{j,k}$ are the Langlands parameters of ϕ_j, and recall that

$$|A_\phi(1,\ldots,1)|^2 = \frac{c_n\langle\phi,\phi\rangle}{L^*(1,\mathrm{Ad}\ \phi)}$$

for some constant c_n, or with the normalization of $\langle\phi,\phi\rangle = 1$,

$$|A_\phi(1,\ldots,1)| = c_n L^*(1,\mathrm{Ad}\ \phi)^{-1/2}.$$

The main idea of the template method will be to replace Φ by an Eisenstein series with the same Langlands parameters and same first coefficient.

Example 17.6. *Consider the case* $3 = 2+1$*. Here let* Φ *be a Maass form for GL(2) with Langlands parameters* $(\alpha_1, -\alpha_1)$*. Then we want to show that*

$$A_{P_{2,1},\Phi}((1,1),s) = cL^*(1, Ad\ \Phi)^{-1/2}L^*(1+3s_1,\Phi)^{-1}$$

up to some constant c. Recall that $s = (s_1, s_2)$ *with* $2s_1 + s_2 = 0$*.*

1. *Replace* Φ *by* $E_{B_2}(*, (\alpha_1, -\alpha_1))$*, which has the same eigenvalues of the invariant differential operators on* $\mathfrak{h}^2$ *as* Φ*; i.e. has the same Langlands parameters. This creates a new Eisenstein series on* $P_{2,1}$ *denoted* $E_{B_3,new}(g, s*)$*.*

2. $E_{B_3,new}$ *has the same Langlands parameters as* $E_{P_{2,1},\Phi}$*, and it has to be a Borel Eisenstein series, because it only involves powers of y! The Langlands parameters of* $E_{B_3,new}$ *are* $(s_1^*, s_2^*, s_3^*) = (\alpha_1 + s_1, -\alpha_1 + s_1, s_2)$*.*

3. *We now use the formula for the first coefficient of the Borel Eisenstein series (up to a constant factor):*

$$\left(\zeta^*(1+s_1^*-s_2^*)\zeta^*(1+s_1^*-s_3^*)\zeta^*(1+s_2^*-s_3^*)\right)^{-1} = \left(\zeta^*(1+2\alpha_1)\zeta^*(1+\alpha_1+3s_1)\zeta^*(1-\alpha_1+3s_1)\right)^{-1}.$$

   Now, we need to take into account that we want the Eisenstein series we replaced Φ *with to have the same first coefficient as* Φ*. The first coefficient of* $E_{B_2}(*, (\alpha_1, -\alpha_1))$ *is* $\zeta^*(1+2\alpha_1)^{-1}$*, but* $|A_\phi(1,1)| = L^*(1, Ad\ \phi)^{-1/2}$ *(up to a constant). Hence we need to scale our factor by*

$$\frac{L^*(1, Ad\ \phi)^{-1/2}}{\zeta^*(1+2\alpha_1)^{-1}}\left(\zeta^*(1+2\alpha_1)\zeta^*(1+\alpha_1+3s_1)\zeta^*(1-\alpha_1+3s_1)\right)^{-1}.$$

   The $\zeta^*(1+2\alpha_1)$ *cancels, and we note that* $\zeta^*(1+\alpha_1+3s_1)\zeta^*(1-\alpha_1+3s_1)$ *appears in the completed L-function for* Φ*. Thus we get the desired result.*

Example 17.7. *Now consider the $4 = 2 + 2$ case. Here we care about*

$$E_{P_{2,2},\phi_1\otimes\phi_2}(g, s);$$

recall that $s = (s_1, s_2)$ with $s_1 + s_2 = 0$. In this case we want to show that the first coefficient is of the form

$$E_{P_{2,2},\Phi} = L^*(1, Ad\ \phi_1)^{-\frac{1}{2}} L^*(1, Ad\ \Phi_2)^{-\frac{1}{2}} L^*(1 + 2s_1, \phi_1 \times \phi_2)^{-1}.$$

1. *Replace ϕ_1, ϕ_2 by $E_{B_2}(*, \alpha_1)E_{B_2}(*, \alpha_2)$, where $(\alpha_1, -\alpha_1)$ are the Langlands parameters for ϕ_1, and $\alpha_2 = (\alpha_2, -\alpha_2)$ are the Langlands parameters for ϕ_2.*
2. *We get Borel Eisenstein series $E_{B_4,new}(g, s^*)$, where $s^* = (s_1 + \alpha_1, s_1 - \alpha_1, -s_1 + \alpha_2, -s_1 - \alpha_2)$.*
3. *Using the formula for the first coefficient of the Borel Eisenstein series, up to a constant, we get first coefficient*

   $$\Big(\zeta^*(1 + 2\alpha_1)\zeta^*(1 + 2\alpha_2) \prod \zeta^*(1 + 2s_1 \pm \alpha_1 \pm \alpha_2)\Big)^{-1},$$

   where the product is taken over all 4 possibilities. This product turns into $L^*(1 + 2s_1, \phi_1 \times \phi_2)^{-1}$, up to some constant.

   Finally, we take into account the normalization, so we multiply by

   $$\frac{L^*(1, Ad\ \phi_1)^{-1/2} L^*(1, Ad\ \phi_2)^{-1/2}}{\zeta^*(1 + 2\alpha_1)^{-1}\zeta^*(1 + 2\alpha_2)^{-1}}.$$

   This cancels the remaining Riemann zeta factors and gives the desired result.

This method generalizes similarly to any Langlands Eisenstein series.

Remark 17.8. *Note that the first coefficient involves Rankin-Selberg L-products – one can use this method to give another proof of the functional equation of Rankin-Selberg L-function (Shahidi-Kim).*

Next time, we will complete the computation for all Fourier coefficients of Eisenstein series towards the Kuznetsov trace formula.

18 Lecture 18 - 4/3/25

18.1 General Fourier Coefficients for Langlands Eisenstein Series

Recall that we defined the normalized Whittaker functions

$$W_\alpha(g) = \prod_{1\le j<k\le n} \frac{\Gamma\left(\frac{1+\alpha_j-\alpha_k}{2}\right)}{\pi^{\frac{1+\alpha_j-\alpha_k}{2}}} \int_{U_n(\mathbb{R})} |w_n u g|_{B_n}^{\alpha+\rho_{B_n}} e^{-2\pi(u_{1,2}+\cdots+u_{n-1,n})}\, \mathrm{d}u\,,$$

where w_n is the long element (1s on the anti-diagonal).
Today, we are interested in computing the Mth Fourier-Whittaker coefficient of $E_{P,\Phi}$, where $M = (m_1,\dots,m_{n-1}) \in \mathbb{Z}^{n-1}$, $P = P_{n_1,\dots,n_r}$, and $\Phi = \phi_1\otimes\cdots\otimes\phi_r$, with ϕ_i a Maass form on $\mathrm{SL}(n_i,\mathbb{Z})$. As we have discussed before, this is determined by the integral

$$\int_0^1 \dots \int_0^1 E(ug,s)\overline{\psi_M(u)}\,\mathrm{d}u = \frac{A_{P,\Phi}(M,s)}{\prod_{k=1}^{n-1} m_k^{\frac{k(n-k)}{2}}} W_{\alpha_{P,\Phi}(s)}(M^* g),$$

where $\alpha_{P,\Phi}(s)$ are the Langlands parameters of $E_{P,\Phi}(s)$ and

$$M^* = \begin{pmatrix} m_1\dots m_{n-1} & & & \\ & \ddots & & \\ & & m_{n-1} & \\ & & & 1\end{pmatrix}.$$

Remark 18.1. *Note that this definition of the matrix differs from Dorian's book because we do not swap the order of the m_i here, unlike in Dorian's book. In addition, note that the normalized Whittaker function makes the coefficients slightly different from Dorian's book.*

Remark 18.2. *One can instead compute this integral directly with the Bruhat decomposition and Kloosterman sums, but that is extremely tedious.*

Here $A_{P,\Phi}(M,s)$ is the Mth Fourier coefficient. Last time, we discussed that

$$A_{P,\Phi}(M,s) = A_{P,\phi}((1,\dots,1),s)\lambda_{P,\Phi}(M,s),$$

where $\lambda_{P,\Phi}(M,s)$ is a Hecke eigenvalue which can be computed via the Hecke operators T_m and the multiplicativity relations.
In particular, we will consider the simplest case

$$\lambda_{P,\Phi}((m,1,\dots,1),s),$$

the eigenvalue of the Hecke operators T_m.

Theorem 18.3. $\lambda_{P,\Phi}((m,1,\dots,1),s) = \sum_{\substack{c_1\cdots c_r=m\\ c_i\in\mathbb{N}}} \prod_{j=1}^r \lambda_{\Phi_j}(c_j)c_j^{s_j}$.

Remark 18.4. *As far as Dorian is aware, the first time this formula appeared was in his book.*

Proof. We apply T_m to the power function times Φ, using that the Hecke operators are Γ_n invariant.
It is enough to compute

$$T_m\Phi(y)|y|_P^{s+\rho} = m^{-\frac{n-1}{2}} \sum_{\substack{c_1\cdots c_n=m\\ c_{i,\ell}<c_\ell}} \prod_{i=1}^r \psi_i(\mathfrak{m}_{n_i}(cy))|\det(\mathfrak{m}_{n_i}(cy))|^{s+\rho},$$

where c is the matrix appearing in the definition of the Hecke operator.
We can then rewrite the sum in terms of r blocks of size n_i on the diagonal; each block of size n_i with determinant c_i, and appearing in the definition of the Hecke operator T_{c_i} for $\mathrm{SL}(n_i,\mathbb{Z})$, so the T_m operator reduces to individual T_{c_i} operators on small blocks on the diagonal. We also need to sum over elements lying above the $n_i\times n_i$ block, which do not affect the Hecke operator computation – these contribute $c_i^{n_1+\cdots+n_{i-1}}$ to the comptutation. Working through everything gives the desired answer. ∎

18.2 Non-vanishing of L-functions via Eisenstein series

This result was first proved by Selberg for GL(2), although unpublished. It was then extended by Jacquet-Shalika to Chevellay groups and general number fields.
We remark that $\zeta(1+it) \neq 0$ for $t \in \mathbb{R}$ is equivalent to the prime number theorem.

Theorem 18.5 (Jacquet-Shalika)**.** *Let ϕ be a Maass form for $SL(n,\mathbb{Z})$ with functional equation for the L-function invovling $s \to 1-s$. Then $L(1+it,\phi) \neq 0$ for all $t \in \mathbb{R}$.*

We state the functional equation for $E_{P,\Phi}(g,s)$; given $\sigma \in S_r$, then for $P = P_{n_1,\ldots,n_r}$, we define $\sigma P = P_{\sigma(1),\ldots,\sigma(r)}$, $\sigma\phi = \phi_{\sigma(1)} \otimes \cdots \otimes \phi_{\sigma(r)}$, and similarly for $s = (s_1,\ldots,s_r)$. Then $E_{P,\Phi}(g,s) = E_{\sigma P,\sigma\phi}(g,\sigma s)$.

Remark 18.6. *Note here that this definition does not normalize the first coefficient to be 1.*

One can also show that for the terms $M = (m_1,\ldots,m_{n-1})$, the functional equation also holds.

Example 18.7. *In the $SL(2,\mathbb{Z})$ case,*

$$E(g,s) = y^{1/2+s_1} + \frac{\zeta^*(2s_1)}{\zeta^*(2s+1)} y^{1/2-s_1} + \frac{2\sqrt{y}}{\zeta^*(2s_1+1)} \sum_{n\neq 0} \frac{\sigma_{2s_1}(n)}{|n|^{s_1}} K_{s_1}(2\pi|n|y) e^{2\pi inx},$$

where $s = (s_1,-s_1)$.
Assume $\zeta(1+it_0) = 0$ for some $t_0 \in \mathbb{R}$. We note that

$$\langle \zeta^*(2s+1)E(*,s),\phi\rangle = 0,$$

as the integral picks off the constant term of ϕ, which is 0.
Now, consider $\zeta^*(1+it_0)E(g,\frac{it_0}{2})$. Using that we have a zero, this means the the constant term is 0, using the fact that $\zeta(1-it_0) = 0$ by conjugation, and using the Riemann zeta functional equation. Moreover, we note that as $y \to \infty$, the Eisenstein series does not vanish and is non-constant. Hence, $\zeta^*(1+it_0)E(g,\frac{it_0}{2})$ is a Maass form Φ.
As we discussed before, $\langle\Phi,\Phi\rangle = 0$. However, this contradicts the assumption that Φ does not vanish. Hence, we get a contradiction.

Remark 18.8. *With this method, Sarnak can get an error term on the prime number theorem.*

One can apply this technique more generally – see Dorian's book for an example on $E_{P_{2,1},\Phi}(g,s)$. In this case, the first coefficient is $L^*(1,\mathrm{Ad}\ \Phi)^{-1/2}L^*(1+3s_1,\Phi)^{-1}$, so everything is multiplied by $L^*(1+3s_1,\Phi)$. We can do something similar, letting $s_1 = it_0/3$.
Next time we will do the Kuznetsov trace formula.

19 Lecture 19 - 4/8/25

The Kuznetsov trace formula is obtained from taking two Poincare series and taking their inner product. One way to evaluate this is by taking the spectral expansion of one of the Poincare series and unraveling the other series, giving the "spectral side" of the trace formula. The other way is to take the Fourier expansion of one of the Poincare series and unravel the other series, giving the "geometric side".

19.1 Poincare Series

Definition 19.1. *Let $P_0 : \mathbb{R} \to \mathbb{C}$ be a test function such that $P_0(y) \ll y^{1+\varepsilon}$ when $0 < y \le 1$, and $\ll y^{-\varepsilon}$ when $1 \le y$. Moreover, letting $u = \begin{pmatrix} 1 & u_1 \\ & 1 \end{pmatrix}$, define $\psi(u) = e^{2\pi i u_1}$. Let $M = \begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix}$ with $m \in \mathbb{Z}$ nonzero. Let $\Gamma = SL(2,\mathbb{Z})$ and Γ_∞ be the Borel subgroup. Then*

$$P^M(g, P_0) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} P(M\gamma g)\psi(M\gamma g),$$

where

$$P\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) = P_0(y)$$

is a function $P : \mathfrak{h}^2 \to \mathbb{C}$.

19.2 Kuznetsov Trace Formula for SL$(2,\mathbb{Z})$

Consider two Poincare series $P^M(g, P_0)$ and $Q^N(g, Q_0)$, with m, n nonzero. We are interested in the inner product

$$\left\langle P^M(*, P_0), Q^N(*, Q_0)\right\rangle = \int_{\Gamma\backslash\mathfrak{h}^2} P^M(g, P_0) Q^N(g, Q_0) \frac{\mathrm{d}x\,\mathrm{d}y}{y^2},$$

where $g = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}$.

As mentioned before, we evaluate this in two ways.

1. We use the spectral expansion of P^M and unravel Q^N, giving the spectral side of the KTF.
2. Use the Fourier expansion of P^M and unravel Q^n, giving the geometric side of the KTF.

Remark 19.2. *The Selberg trace formula is similar - breaking up $SL(2,\mathbb{Z})$ via conjugacy classes or double cosets to get two sides.*

19.3 Spectral side

Let $\{\eta_j\}_{j=1,2,3,\ldots}$ be an orthonormal basis of Maass forms for SL$(2,\mathbb{Z})$. Then the spectral decomposition tells us

$$P^M(g, P_0) = \sum_{j=0}^{\infty} \left\langle P^M(*, P_0), \eta_j\right\rangle \eta_j(g) + \frac{1}{4\pi i}\int_{\mathrm{Re}(s_1)=0} \left\langle P^M(*, P_0), E(*, s)\right\rangle E(g, s)\,\mathrm{d}s_1$$

where here $s = (s_1, -s_1)$ are Langlands parameters, explaining why the integral is over $\mathrm{Re}(s_1) = 0$ (rather than 1/2).

Hence, we get that

$$\begin{aligned}&\left\langle P^M(g, P_0), Q^N(g, Q_0)\right\rangle \\ =&\sum_{j=0}^{\infty} \left\langle P^M(*, P_0), \eta_j\right\rangle \left\langle \eta_j(g), Q^N(g, Q_0)\right\rangle + \frac{1}{4\pi i}\int_{\mathrm{Re}(s)=0} \left\langle P^M(*, P_0), E(*, s)\right\rangle \left\langle E(*, s), Q^N(*, Q_0)\right\rangle \mathrm{d}s_1\end{aligned}$$

To understand this, we need to understand

$$\langle P^M(*, P_0), \eta_j \rangle = \int_{\Gamma\backslash\mathfrak{h}^2} P^M(g, P_0)\overline{\eta_j(g)}\frac{\mathrm{d}x\,\mathrm{d}y}{y^2}.$$

Using the summation definition of $P^M(g, P_0)$ and making the change of variables $g \to \gamma^{-1}(g)$, we get

$$\begin{aligned}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\int_{\gamma^{-1}\cdot(\Gamma\backslash\mathfrak{h}^2)} P_0(Mg)\psi(Mg)\frac{\mathrm{d}x\,\mathrm{d}y}{y^2} &= \sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\int_{\gamma^{-1}\cdot(\Gamma\backslash\mathfrak{h}^2)} P_0(Mg)\psi(Mg)\frac{\mathrm{d}x\,\mathrm{d}y}{y^2} \\ &= \int_{y=0}^{\infty}\int_{x=0}^{1} P_0(My)e^{2\pi imx}\overline{\eta_j\left(\begin{pmatrix} y & x \\ 0 & 1\end{pmatrix}\right)}\frac{\mathrm{d}x\,\mathrm{d}y}{h^2} \\ &= \begin{cases} 0 & j=0 \\ \int_{y=0}^{\infty}\int_0^1 e^{2\pi imx}\overline{\eta_j(x+iy)}\,\mathrm{d}x\,P_0(my)\frac{\mathrm{d}y}{y^2}, \end{cases}\end{aligned}$$

as the integral over x picks off the mth coefficient of η_j, and m is nonzero. The $j > 0$ term gives

$$\overline{A_j(m)}\int_0^\infty \sqrt{y}K_{ir_j}(2\pi my)P_0(my)\frac{\mathrm{d}y}{y^2}.$$

Here $A_j(m)$ is the mth coefficient of η_j, which splits as $A_j(1)\lambda_j(m)$. We also need $\langle P^M, E(*, s)\rangle$, which evaluates as

$$\overline{A(m, s_1)}\sqrt{m}\int_0^\infty P_0(y)K_{s_1}(2\pi y)\frac{\mathrm{d}y}{y^{3/2}}$$

where the $\sqrt{m}$ comes from a change of variable. Here let

$$P_0^\sharp(ir_j) := \int_0^\infty P_0(y)K_{s_1}(2\pi y)\frac{\mathrm{d}y}{y^{3/2}},$$

where η_j has Langlands parameter $(ir_j, -ir_j)$ and Laplace eigenvalue $\frac{1}{4} + r_j^2 = \lambda_j$. Plugging these equations in the spectral side gives

$$\langle P^M(*, P_0), Q^N(*, Q_0)\rangle = \sqrt{mn}\sum_{j=1}^{\infty} A_j(m)\overline{A_j(n)}P_0^\sharp(ir_j)\overline{Q_0^\sharp(ir_j)} + \frac{\sqrt{mn}}{4\pi}\int_{-i\infty}^{i\infty} A(m, s_1)\overline{A(n, s_1)}P_0^\sharp(s_1)\overline{Q_0^\sharp(s_1)}\,\mathrm{d}s_1\,.$$

Remark 19.3. *Note that $SL(2,\mathbb{Z})$ Maass forms are self-dual (in particular, the Hecke operators are self-adjoint) to get that the Hecke eigenvalues λ_j are real. Hence, some of the conjugates on the right hand side can be simplified.*

19.4 Geometric Side

We move to the geometric side, which is harder to compute.
The classical version of the Poincare series is given by

$$P^M(g, P_0) = P_0(m)e^{2\pi imx} + \sum_{c=1}^{\infty}\sum_{\substack{d\in\mathbb{Z} \\ (c,d)=1}} P_0\left(\frac{my}{|cz+d|^2}\right)e^{2\pi im\mathrm{Re}\left(\frac{az+b}{cz+d}\right)},$$

where here a and b are chosen such that $\begin{pmatrix} a & b \\ c & d\end{pmatrix} = \mathrm{SL}(2,\mathbb{Z})$. However, for each choice of (c, d), the summation does not depend on the choice of (a, b).
On the RHS, there will be Kloosterman sums

$$S(m, n; c) = \sum_{\substack{a=1 \\ (a,c)=1 \\ ad\equiv 1 \pmod{c}}}^{c} e^{2\pi i\left(\frac{am+nd}{c}\right)}.$$

Remark 19.4. *Andre Weil proved the bound of $\ll c^{1/2+\varepsilon}$ as an application of the Riemann Hypothesis for curves – this is why this side is known as the "geometric side".*

We can compute the nth Fourier coefficient of P^M:

Lemma 19.5.

$$\int_0^1 P^M(g,P_0)e^{-2\pi inx}\,\mathrm{d}x = \delta_{m,n}P_0(my) + y\sum_{c=1}^{\infty} S(m,n;c)\int_{-\infty}^{\infty} P_0\left(\frac{my}{c^2y(x^2+1)}\right)e^{-2\pi ix\left(\frac{m}{c^2y(x^2+1)}+ny\right)}\,\mathrm{d}x\,.$$

Sketch. We have that

$$\int_0^1 P^M(g,P_0)e^{-2\pi inx}\,\mathrm{d}x = \delta_{m,n}P_0(my) + \int_0^1 \sum_{c=1}^{\infty}\sum_{\substack{d\in\mathbb{Z}\\ \gcd(c,d)=1}} P_0\left(\frac{my}{|cz+d|^2}\right)e^{2\pi im\mathrm{Re}\left(\frac{az+b}{cz+d}\right)}e^{-2\pi inx}\,\mathrm{d}x\,,$$

then computation is done. More details can be found in Dorian's trace formula notes on his website. □

We now apply this formula and unravel Q. Let

$$I(y) = y\sum_{c=1}^{\infty} S(m,n;c)\int_0^{\infty} P_0\left(\frac{my}{c^2y(x^2+1)}\right)e^{-2\pi ix\left(\frac{m}{c^2y(x^2+1)}+ny\right)}\,\mathrm{d}x\,.$$

Specifically, unraveling gives

$$\begin{aligned}\left\langle P^M(*,P_0),Q^N(*,Q_0)\right\rangle &= \int_{y=0}^{\infty}\int_{x=0}^{1} P^M(g,P_0)\overline{Q_0(ny)}e^{-2\pi inx}\frac{\mathrm{d}x\,\mathrm{d}y}{y^2}\\ &= \int_0^{\infty}(\delta_{m,n}P_0(my)+I(y))\overline{Q_0(ny)}\frac{\mathrm{d}y}{y^2}.\end{aligned}$$

Hence, the Kuznetsov trace formula comes out as

$$\begin{aligned}&\sqrt{mn}\sum_{j=1}^{\infty}A_j(m)\overline{A_j(n)}P_0^{\sharp}(ir_j)\overline{Q_0^{\sharp}(ir_j)} + \frac{\sqrt{mn}}{4\pi}\int_{-i\infty}^{i\infty}A(m,s_1)\overline{A(n,s_1)}P_0^{\sharp}(s_1)\overline{Q_0^{\sharp}(s_1)}\,\mathrm{d}s_1\\ =&\delta_{mn}\int_0^{\infty}P_0(my)\overline{Q_0(ny)}\frac{\mathrm{d}y}{y^2} + \sum_{c=1}^{\infty}S(m,n;c)\int_0^{\infty}\int_{-\infty}^{\infty}P_0\left(\frac{m}{c^2y(x^2+1)}\right)\overline{Q_0(ny)}e^{-2\pi ix\left(\frac{m}{c^2y(x^2+1)}+ny\right)}\frac{\mathrm{d}x\,\mathrm{d}y}{y}.\end{aligned}$$

Here we note that the main term is the first term on the right hand side.

19.5 Application of Kuznetsov Trace Formula to Weyl's Law

We want to show that

$$\sum_{\lambda_j\le T} 1 \sim cT.$$

We will end up showing that

$$\sum_{\lambda_j\le T}\frac{1}{L(1,\mathrm{Ad}\ \phi_j)} \sim cT,$$

and a method of Iwaniec will show that this is equivalent.

Remark 19.6. *We can remove the adjoint L-functions for $GL(2)$ and $GL(3)$, but not in general.*

Choose $m=n=1$. Then $|A_j(1)|^2 = \frac{c}{L^*(1,\mathrm{Ad}\ \eta_j)} = \frac{c}{L(1,\mathrm{Ad}\ \eta_j)\Gamma\left(\frac{1+2ir_j}{2}\right)\Gamma\left(\frac{1-2ir_j}{2}\right)}$.

Recall that

$$P_0^{\sharp}(ir) = \int_0^{\infty}P_0(y)K_{ir}(2\pi y)\frac{\mathrm{d}y}{y^{3/2}}.$$

We have inverse transform (the Kontorovich-Lebedev transform)

$$P_0(y) = \frac{1}{\pi}\int_{-\infty}^{\infty} P_0^\sharp(ir)\sqrt{y}K_{ir}(2\pi y)\frac{\mathrm{d}r}{\Gamma(ir)\Gamma(-ir)}$$

Let $R \geq 5$ and $T \to \infty$. Consider test function of the form

$$P_{T,R}^\sharp(W) = e^{-\frac{W^2}{2T^2}}\Gamma\left(\frac{2+R+2W}{4}\right)\Gamma\left(\frac{2+R-2W}{4}\right).$$

Note that by Stirling's formula,

$$\frac{\Gamma\left(\frac{2+R+2W}{4}\right)\Gamma\left(\frac{2+R-2W}{4}\right)}{\Gamma\left(\frac{1+2ir_j}{2}\right)\Gamma\left(\frac{1-2ir_j}{2}\right)} \sim \sqrt{2\pi}r_j^R$$

so on the spectral side, the main term becomes

$$\sum_{j=1}^{\infty}|A_j(1)|^2|P_{T,R}^\sharp(ir_j)|^2 = \sqrt{2\pi}e^{1/4}c\sum_{j=1}^{\infty}e^{-\lambda_j/T^2}\frac{\lambda_j^R}{L(1,\mathrm{Ad}\ \eta_j)} \sim c_0\sum_{\lambda\leq T^2}\frac{1}{L(1,\mathrm{Ad}\ \eta_j)},$$

where the approximation comes from partial summation and from using the fact that $e^{-\lambda_j/T^2}$ roughly counts λ up to T^2. The main error term comes from the Eisenstein series, the error term on the spectral side.
To evaluate the main term on the geometric side, we will need to use Plancherel's formula. We will do this next time.

20 Lecture 20 - 4/10/25

20.1 Main Term in GL(2) Kuznetsov Trace Formula

Recall that we had a smooth test function $P_0 : \mathbb{R}_{\geq 0} \to \mathbb{C}$ with the growth condition $|P_0(y)| \ll y^{1+\varepsilon}$ if $0 < y \leq 1$ and y^{-B} if $1 < y$. We also have the Kontorovich-Lebedev transform

$$P_0^\sharp(ir) := \int_0^\infty P_0(y)\sqrt{y}K_{ir}(2\pi y)\frac{\mathrm{d}y}{y^2}$$

and the inverse transform

$$P_0(y) = \frac{1}{\pi}\int_{-\infty}^\infty P_0^\sharp(ir)\sqrt{y}K_{ir}(2\pi y)\frac{\mathrm{d}r}{|\Gamma(ir)|^2}.$$

Last time, we showed that the main term M of the Kuzentsov trace formula was

$$M = m\int_0^\infty |P_0(y)|^2\frac{\mathrm{d}y}{y^2}.$$

Theorem 20.1 (Plancherel)**.**

$$\int_0^\infty |P_0(y)|^2\frac{\mathrm{d}y}{y^2} = \frac{1}{\pi}\int_{-\infty}^\infty |P_0^\sharp(ir)|^2\frac{\mathrm{d}r}{|\Gamma(ir)|^2}.$$

Proof.

$$\begin{aligned}\int_0^\infty |P_0(y)|^2\frac{\mathrm{d}y}{y^2} &= \int_{y=0}^\infty P_0(y)\int_{r=-\infty}^\infty \frac{1}{\pi}\overline{P_0^\sharp(ir)}\sqrt{y}K_{ir}(2\pi y)\frac{\mathrm{d}r}{|\Gamma(ir)|^2}\frac{\mathrm{d}y}{y^2}\\ &= \frac{1}{\pi}\int_{r=-\infty}^\infty \overline{P_0^\sharp(ir)}\int_{y=0}^\infty P_0(y)\sqrt{y}K_{ir}(2\pi y)\frac{\mathrm{d}y}{y^2}\frac{\mathrm{d}r}{|\Gamma(ir)|^2}\\ &= \frac{1}{\pi}\int_{r=-\infty}^\infty \overline{P_0^\sharp(ir)}P_0^\sharp(ir)\frac{\mathrm{d}r}{|\Gamma(ir)|^2},\end{aligned}$$

as desired, where we use that $K_{ir} = K_{-ir}$. ☐

Hence to choose a test function P_0, it is sufficient to consider the conditions on $P_0^\sharp$. Last time, we chose

$$P_{T,R}^\sharp(ir) := e^{-\frac{r^2}{2T^2}}\left|\Gamma\left(\frac{2+R+2ir}{4}\right)\right|^2,$$

where $T \to \infty$ and $r \geq 5$. Then the main term in this case is

$$\begin{aligned}M &= \frac{m}{\pi}\int_{-\infty}^\infty |P_{T,R}^\sharp(ir)|^2\frac{\mathrm{d}r}{|\Gamma(ir)|^2}\\ &= \frac{2m}{\pi}\int_0^\infty e^{-\frac{r^2}{T^2}}\frac{\left|\Gamma\left(\frac{2+R+2ir}{4}\right)\right|^4}{|\Gamma(ir)|^2}\,\mathrm{d}r\,.\end{aligned}$$

When $r \gg T$, we get exponential decay, so the main contribution comes from $r \sim T$. Applying Stirling's formula

$$|\Gamma(\sigma+it)| \sim \sqrt{2\pi}|t|^{\sigma-1/2}e^{-\frac{\pi}{2}|t|}$$

gives the asymptotic as $T \to \infty$

$$\begin{aligned}M &\sim \frac{2m}{\pi}\int_0^\infty e^{-\frac{r^2}{T^2}}\frac{((1+r)^{R/4})^4}{((1+r)^{-1/2})^2}\,\mathrm{d}r\\ &\sim \frac{2m}{\pi}\int_0^\infty e^{-\frac{r^2}{T^2}}(1+r)^{R+1}\,\mathrm{d}r\\ &\sim T^{R+2}\frac{2m}{\pi}\int_0^\infty e^{-r^2}\left(\frac{1}{T}+r\right)^{R+1}\,\mathrm{d}r\\ &\sim c_0T^{R+2}\end{aligned}$$

where the final integral goes to a constant as $T \to \infty$ and the transformation $r \to rT$ is used. TODO: This calculation doesn't seem quite right.
From the spectral side, this is equivalent to a summation of the form $\sum_{\lambda \le T} a_\lambda \lambda^R$. Via Abel partial summation, one can transform the this into information about $\sum_{\lambda \le T} a_\lambda$, and it turns out to be asymptotic to T^2; i.e. the dependence on R goes away.

20.2 Orthogonality for Fourier Coefficients of Maass Forms

We spend some time discussing the history of results in this direction.

- In 1837, Dirichlet showed an orthogonality relation for Dirichlet characters $\chi \pmod q$

$$\frac{1}{\phi(q)} \sum_{\chi} \chi(m)\overline{\chi}(a) = \delta_{a \equiv m \pmod q}.$$

  One can think of Dirichlet characters as GL(1) automorphic forms/representations. Naturally, one can ask if it is possible to extend this to higher GL(n).

- The first result in this direciton is from Bruggeman 1978:

$$\lim_{T \to \infty} \frac{4\pi^2}{T} \sum_{j=1}^{\infty} \frac{\lambda_j(m)\overline{\lambda_j(n)}}{\cosh(\pi r_j)} e^{-\lambda_j/T} = \delta_{m=n},$$

  where $\lambda_j(m)$ is the mth Fourier coefficient of an SL$(2,\mathbb{Z})$ Maass form ϕ_j with Langlands parameter $(ir_j, -ir_j)$ and laplace eigenvalue $\lambda_j = \frac{1}{4} + r_j^2$.

  For GL(2) or higher, this is an infinite sum, so we need an exponential decay term for convergence.

- In 1984, Sarnak showed a similar result in his paper *Statistical properties of eigenvalues of Hecke operators*.

- In 1997, both Conrey-Duke-Farmer (*Distribution of Hecke eigenvalues*) and J.P. Serre showed a result for holomorphic modular forms (*Repartition asymptotique des valeurs de l'operateur de Hecke*) – these were papers talking about the vertical Sato-Tate conjecture (fixing a prime, and averaging over a family of modular forms and seeing how a_p varies as the family changes), so these kind of orthogonality relations come into play.

- The next result is from Fan Zhou, a student of Dorian's:

  Conjecture 20.2. *Consider an orthonormal basis of Maass forms $\{\phi_j\}_{j=1,2,\ldots}$ for SL$(n,\mathbb{Z})$, and each ϕ_j has Langlands parameters $\alpha^{(j)} = (\alpha_1^{(j)}, \ldots, \alpha_n^{(j)})$ with Hecke eigenvalues $\lambda_j(k)$, for $k = 1, 2, 3, \ldots$. Let $\mathcal{L}_j = L(1, Ad\ \phi_j)$ be adjoint L-functions, and let $h_T : \mathbb{C}^n \to \mathbb{C}$ be a smooth test function with support on eigenvalues $0 < \lambda(\phi_j) \ll T$.*

  Then

$$\frac{\sum_{j=1}^{\infty} \lambda_j(\ell)\overline{\lambda_j(m)} \frac{h_T(\alpha^{(j)})}{\mathcal{L}_j}}{\sum_{j=1}^{\infty} \frac{h_T(\alpha^{(j)})}{\mathcal{L}_j}} = \delta_{\ell=m} + O(T^{-\theta}),$$

  for some $\theta > 0$.

- In the same year, the conjecture was proven for the first time for SL$(3,\mathbb{Z})$ by Kontorovich and Dorian:

$$\sum \frac{\lambda_j(m)\overline{\lambda_j(\ell)} \frac{h^T(\alpha(j))}{\mathcal{L}_j}}{\sum_{j=1}^{\infty} \frac{h^T(\alpha(j))}{\mathcal{L}_j}} = \delta_{m=\ell} + O(m\ell^2 T^{-2+\varepsilon}).$$

  This h^T ends up being up some $P^{\sharp}_{T,R}$ function. Simulatenously, a result like this was proven by Blomer.

Remark 20.3. *Applications of this are vertical Sato-Tate and Weyl's law with a power-saving error term. It is possible to get rid of the $\mathcal{L}_j$ weighting for $SL(2,\mathbb{Z})$ and $SL(3,\mathbb{Z})$, but it is unknown how to do this for $SL(4,\mathbb{Z})$ or higher. Conjecturally, $\lambda_j^{-\varepsilon} \ll \mathcal{L}_j \ll \lambda_j^{\varepsilon}$; the lower bound is the bound that is difficult to prove.*

- In 2014, Blomer, Buttcane, and Raulf got improvements on the orthogonal relation in $\mathrm{SL}(3,\mathbb{Z})$.
- How about for $\mathrm{SL}(4,\mathbb{Z})$? Stade-Woodbury-Dorian proved Fan Zhou's conjecture for $\mathrm{SL}(4,\mathbb{Z})$ in 2021.
- Also in 2021, Matz-Templier, Matz-Finis proved Zhou's conjecture for all $\mathrm{SL}(n,\mathbb{Z})$ with $n \geq 2$ using the Selberg trace formula, but with worse error term. The residual spectrum appears in the computation for Selberg trace formula, but not in the Kuznetsov trace formula, as the Poincare series are orthogonal to the residual spectrum.
- In 2024, Stade-Woodbury-Dorian, Nelson-Jana, Blomer all have results for all $\mathrm{GL}(n)$ using the KTF. Next time, Dorian will give the main idea of the approach in his paper. Nelson-Jana/Blomer avoid contributions by the continuous spectrum by a theorem of Kazhdan. The error terms are all very weak.
- Stade-Woodbury-Dorian prove Zhou's conjecture with very strong power savings for $\mathrm{SL}(4,\mathbb{Z})$ and $\mathrm{SL}(5,\mathbb{Z})$. For $\mathrm{SL}(n,\mathbb{Z})$, for $n > 5$, they prove Zhou's conjecture (with power-savings error term) with two conjectures:
  - Lower bound conjecture for $L(s,\phi_j \times \phi_\ell)$ on $\mathrm{Re}(s) = 1$. If ϕ_j has Langlands parameters $\alpha^{(j)}$ with analytic conductor
    $$c(\phi_j) = \prod_{i=1}^{n}(1 + |\alpha_i^{(j)}|),$$
    then we want
    $$L(1+it,\phi_j \times \phi_\ell) \gg (c(\phi_j)c(\phi_\ell))^{-\varepsilon-1}(1+|t|)^{-\varepsilon_2}.$$
    Remark 20.4. *This appears in the Eisenstein series contribution in the trace formula, and can be proven up to $SL(5,\mathbb{Z})$. If we knew $\phi_j \times \phi_\ell$ was automorphic on $SL(j\times\ell,\mathbb{Z})$ (a conjecture of Langlands), then this could be proven.*

    Remark 20.5. *Qiao Zhang (another student of Dorian's) proved that for ϕ on $SL(n,\mathbb{Z})$ and ϕ' on $SL(n',\mathbb{Z})$,*
    $$L(1+it,\phi\times\phi') \gg (c(\phi)c(\phi'))^{-\theta_{n,n'}}(2+|t|)^{-\frac{nn'}{2}\left(1-\frac{1}{n+n'}\right)\varepsilon}$$
    where $\theta_{n,n'} = n + n' + \varepsilon$.

    If one can improve this by dividing the exponents by a factor of 2, then Dorian's result would hold.
  - Ishi-Stade conjecture: $\int_0^\infty \sqrt{y}K_{ir}(y)y^s\frac{dy}{y} \sim \Gamma\left(\frac{s+ir}{2}\right)\Gamma\left(\frac{s-ir}{2}\right)$ satsifies a functional equation $s \to s+1$. The conjecture is that a functional equation of this type holds for all $\mathrm{SL}(n,\mathbb{Z})$ Whittaker functions. For $\mathrm{SL}(3,\mathbb{Z})$, you get a product of six gamma factors over one gamma, and this conjecture holds. This is proven up to $\mathrm{SL}(7,\mathbb{Z})$.

Next time, we will do the Kuznetsov trace formula for $\mathrm{SL}(n,\mathbb{Z})$, both the spectral and main term.

21 Lecture 21 - 4/15/25

I am very busy, so these notes will be unedited and contain more mistakes than usual. I hope to get them edited by the end of May.

21.1 Whittaker Transform

This will be the generalization of the Kontorovich-Lebedev transform for GL(n).

Definition 21.1. *The* ***vector space of Langlands parameters*** *is denoted*

$$\mathcal{H}^n := \{\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n \mid \alpha_1 + \cdots + \alpha_n = 0\}.$$

We once again redefine the Whittaker functions; Dorian is revising his book which leads to small revisions in definitions.

Definition 21.2. *Let $\alpha \in \mathcal{H}^n$. Then for $g \in \mathfrak{h}^n$, with Iwasawa decomposition $g = xy$, the* ***normalized Whittaker function for SL***$(n, \mathbb{Z})$ *is defined by*

$$W_\alpha^{\pm}(g) = \prod_{1 \le i < j \le n} \frac{\Gamma\left(\frac{1+\alpha_j - \alpha_k}{2}\right)}{\pi^{\frac{1+\alpha_j-\alpha_k}{2}}} \int_{U_n(\mathbb{R})} |w_n u g|_B^{\alpha+\rho_B} \overline{\psi_{1,\dots,1,\pm 1}}(u)\, \mathrm{d}u\,.$$

This function is invariant under any permutation of Langlands parameters – i.e. it satisfies a functional equation. Here we use + for the even Maass forms and − for odd Maass forms.

Remark 21.3. *If g is a diagonal matrix, we can assume a + sign; only the x-coordinates matters for the ± sign. In that case, we drop the ± sign.*

Proposition 21.4. *Let $f : \mathbb{R}_{>0}^{n-1} \to \mathbb{C}$. Then we define the* ***Whittaker transform*** *$f^\sharp(\mathcal{H}^n) \to \mathbb{C}$ by*

$$f^\sharp(\alpha) = \int_{y_1=0}^{\infty} \cdots \int_{y_{n-1}=0}^{\infty} f(y) W_\alpha(y) \prod_{k=1}^{n-1} \frac{\mathrm{d}y_k}{y_k^{k(n-k)+1}},$$

with inverse transform

$$f(y) = \frac{1}{\pi^{n-1}} \int_{\substack{\alpha \in \mathcal{H}^n \\ Re(\alpha_j)=0}} \frac{f^\sharp(\alpha)\overline{W_\alpha(y)}}{\prod_{1 \le k \ne \ell \le n} \Gamma\left(\frac{\alpha_k - \alpha_\ell}{2}\right)}\, \mathrm{d}\alpha\,.$$

Proof. Proof can be found in Dorian's paper with Alex Kontorovich. □

21.2 Poincare Series for SL($n, \mathbb{Z}$)

Remark 21.5. *These definitions may vary based on source – could include/not include test function function, power function, character.*

The ingredients we'll want:

- $M = (m_1, \dots, m_{n-1}) \in \mathbb{Z}^{n-1}$ and $M^* = \begin{pmatrix} m_1 m_2 \dots m_{n-1} & & & & \\ & m_1 m_2 \dots m_{n-2} & & & \\ & & \ddots & & \\ & & & m_1 & \\ & & & & 1 \end{pmatrix}$.
- $\psi_M(u) = e^{2\pi i (m_1 u_{1,2} + \cdots + m_{n-1} u_{n-1,n})}$
- A smooth test function $p : \mathfrak{h}^n \to \mathbb{C}$ satsifying $p(xy) = p(y)$.

Definition 21.6.

$$P^M(g) = \sum_{\gamma \in U_n(\mathbb{Z}) \backslash SL(n,\mathbb{Z})} p(M^* \gamma g) \psi_M(\gamma g),$$

where here $\psi_M(xy) = \psi_M(x)$.

21.3 Spectral Expansion of SL$(n, \mathbb{Z})$ Kuznetsov Trace Formula

Let $F \in \mathcal{L}^2(\mathrm{SL}(n,\mathbb{Z})\backslash\mathfrak{h}^n)$, orthogonal to the constant function. Then we have

$$F(g) = \sum_{j=1}^{\infty} \langle F, \phi_j \rangle \, \phi_j(g) + \sum_{\substack{n=n_1+\cdots+n_r \\ P=P_{n_1,\dots,n_r}}} \sum_{\Phi=\phi_{i_1}\otimes\cdots\otimes\phi_{i_r}} c_P \int_{\substack{n_1s_1+\cdots+n_rs_r=0 \\ \mathrm{Re}(s_j)=0}} \langle F, E_{P,\Phi}(*,s)\rangle \, E_{P,\Phi}(g,s)\,\mathrm{d}s\,,$$

where the ϕ_j are an orthonormal basis of Maass forms.
We need to compute

$$\left\langle P^M, \phi_j \right\rangle = \int_{\Gamma_n\backslash\mathfrak{h}^n} P^M(g)\overline{\phi_j(g)}\,\mathrm{d}g\,.$$

Assume that M is chosen such that none of the m_is are 0; hence, there is no contribution from the residual spectrum (i.e. there is no residual spectrum in the spectral decomposition of the Poincare series).
Unraveling the sum gives

$$\int_{U_n(\mathbb{Z})\backslash\mathfrak{h}^n} P(M^*y)\psi_M(x)\overline{\phi_j(xy)} \prod_{1\le i<j\le n} \mathrm{d}x_{i,j} \prod_{k=1}^{n-1} \frac{\mathrm{d}y_k}{y_k^{k(n-k)+1}}.$$

Note that we can view split the integral, and the integral over x correspond to picking off the Mth coefficient of ϕ_j, as we know that we have Fourier expansion

$$\phi_j(g) = \sum_{\gamma\in U_{n-1}(\mathbb{Z})\backslash\Gamma_{n-1}(\mathbb{Z})} \sum_{m_1=1}^{\infty}\cdots\sum_{m_{n-2}=1}^{\infty}\sum_{m_{n-1}\neq 0} \frac{A_{\phi_j}(M)}{\prod_{k=1}^{n-1} m_k^{k(n-k)/2}} W_\alpha^{\mathrm{sign}(m_{n-1})}\left(M^*\begin{pmatrix}\gamma & \\ & 1\end{pmatrix} g\right).$$

Hence

$$\left\langle P^M, \phi_j \right\rangle = \overline{A_{\phi_j}(M)} \left(\prod_{k=1}^{n-1} m_k^{\frac{k(n-k)}{2}}\right) P^\sharp(\alpha^{(j)}),$$

where $\alpha^{(j)}$ are the Langlands parameters of ϕ_j; along the way there will be a change of variables $y_i \mapsto y_i/m_i$.
Similarly,

$$\left\langle P^M, E_{P,\Phi}(*,s) \right\rangle = \overline{A_{P,\Phi}(M,s)} \left(\prod_{k=1}^{n-1} m_k^{\frac{k(n-k)}{2}}\right) P^\sharp(\alpha_s)$$

where α_s is the Langlands parameter of $E_{P,\Phi}(s)$.
Running through the computation gives

$$\left\langle P^M, P^M \right\rangle = * \sum |A_{\phi_j}(M)|^2 |P^\sharp(\alpha^{(j)})|^2,$$

which we can write as $C + E$, where C is what we want for the orthogonal relation and E is the continuous spectrum. Looking at the Fourier expansion will give $M + K$, where M is the main term and K is the geometric term (primarily Kloosterman sums). This will give the Kuznetsov trace formula.

Remark 21.7. *This gives better results than the Arthur-Selberg trace formula because they have to deal with the residual spectrum.*

21.4 Kloosterman sums for SL$(n, \mathbb{Z})$

Now, we'll need to deal with the Kloosterman sums. We need the Bruhat decomposition for this:

Definition 21.8. *Let $n \ge 2$. Then we have the **Bruhat decomposition***

$$GL(n,\mathbb{R}) = B_n(\mathbb{R})W_nB_n(\mathbb{R}),$$

where $B_n(\mathbb{R}) = \begin{pmatrix} * & & & * \\ & * & & \\ & & \ddots & \\ & & & * \end{pmatrix}$ *and W_n is the Weyl group of $GL(n,\mathbb{R})$.*

Proof. One can iteratively construct the decomposition using row/column operations corresponding to $B_n(\mathbb{R})$. ☐

One can explicitely describe the decomposition, due to Friedberg:

Proposition 21.9. *Every $g \in GL(n,\mathbb{R})$ can be written in the form $g = u_1 c w u_2$, where*

$$c = \begin{pmatrix} \varepsilon/c_{n-1} & & & & \\ & c_{n-1}/c_{n-2} & & & \\ & & \ddots & & \\ & & & c_2/c_1 & \\ & & & & c_1 \end{pmatrix},$$

with $w \in W_n$, $\varepsilon = \det(w)\det(g)$, and $u_1, u_2 \in U_n$.

Remark 21.10. *One must put stronger conditions on u_1 and u_2 for this to be unique (i.e. $u_1 \in \Gamma_w$ and u_2 in the complement, which we define later).*

Definition 21.11 (Kloosterman sums for $\mathrm{SL}(2,\mathbb{Z})$)**.** *For $SL(2,\mathbb{Z})$, we have the* ***Kloosterman sum***

$$S(m,n;c) = \sum_{\substack{a=1 \\ (a,c)=1}}^{c} e^{2\pi i\left(\frac{am+\overline{a}n}{c}\right)}.$$

We have the Weil bound of $\ll c^{1/2} + \varepsilon$ - equivalent to RH for an elliptic curve.
We'll use the following notation: let $\Gamma_n = \mathrm{SL}(n,\mathbb{Z})$. For $w \in W_n$, let $\Gamma_w = (w^{-1}(U_n(\mathbb{Z}))^T w) \cap U_n(\mathbb{Z})$. $(w^{-1}(U_n(\mathbb{Z}))^T w$ will correspond to matrices with 1 on diagonal, with some elements on the top still there, over all possible rows/columns). Also, let $G_w = U_n w D_n U_n$, where D_n is the set of diagonal matrices.

Definition 21.12 (Kloosterman Sum)**.**

$$S_w(\psi,\psi',c) := \sum_{\substack{\gamma \in U_n(\mathbb{Z})\backslash(\Gamma_n \cap G_n)/\Gamma_w \\ \gamma = b_1 c w b_2}} \psi(b_1)\psi'(b_2).$$

Example 21.13. *In the $SL(2,\mathbb{Z})$ case, let $c = \begin{pmatrix} c_1^{-1} & 0 \\ 0 & c_1 \end{pmatrix}$, $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and $\Gamma_w = \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix} = U_2(\mathbb{Z})$. Let $b_1 = \begin{pmatrix} 1 & b_1'/c_1 \\ & 1 \end{pmatrix}$ and $b_2 = \begin{pmatrix} 1 & b_2'/c_1 \\ & 1 \end{pmatrix}$. Then for any $\gamma \in SL(2,\mathbb{Z})$, we have the Bruhat decomposition*

$$\gamma = b_1 c w b_2,$$

where $b_1' b_2' \equiv 1 \pmod{c_1}$. In this case,

$$S_w(\psi,\psi',c) = \sum_{b_1' \pmod{c}} \psi(b_1)\psi'(b_2),$$

where we choose M and N for our characters.

Next time we'll continue the computation of the geometric side of the KTF.

22 Lecture 22 - 4/17/25

I am very busy, so these notes will be unedited and contain more mistakes than usual. I hope to get them edited by the end of May.

22.1 Kloosterman Sums

Recall last time we defined Kloosterman sums for SL(n): Letting

$$c = \begin{pmatrix} \frac{1}{c_{n-1}} & & & & \\ & \frac{c_{n-1}}{c_{n-2}} & & & \\ & & \ddots & & \\ & & & \frac{c_2}{c_1} & \\ & & & & c_1 \end{pmatrix},$$

$G_w = U_n w D_N U_n$, and $\Gamma_w = (w^{-1}U_n(\mathbb{Z})^T w) \cap U_n(\mathbb{Z})$, we have that

$$S_w(\psi, \psi', c) = \sum_{\substack{\gamma = b_1 c w b_2 \\ \gamma \in U_n(\mathbb{Z})\backslash \Gamma_n \cap G_w / \Gamma_w}} \psi(b_1)\psi'(b_2).$$

Example 22.1 ($SL(3,\mathbb{Z})$ Kloosterman sum)**.** *Let* $c = \begin{pmatrix} \frac{1}{c_2} & & \\ & \frac{c_2}{c_1} & \\ & & c_1 \end{pmatrix}$*, and* $w = \begin{pmatrix} & & -1 \\ & 1 & \\ 1 & & \end{pmatrix}$*. TODO: is the minus sign supposed to be there? For the long element,* $\Gamma_w = U_3(\mathbb{Z})$*. Let*

$$b_1 = \begin{pmatrix} 1 & \alpha_2 & \alpha_3 \\ & 1 & \alpha_1 \\ & & 1 \end{pmatrix}$$

and

$$b_2 = \begin{pmatrix} 1 & \beta_2 & \beta_3 \\ & 1 & \beta_1 \\ & & 1 \end{pmatrix},$$

both in $U_3(\mathbb{Q})$*, and let* $\gamma = (\gamma_{i,j})_{1 \le i,j \le 3} \in SL(3,\mathbb{Z})$*. Then consider any* $\gamma \in U_3(\mathbb{Z})\backslash SL(3,\mathbb{Z}) \cap G_w / U_3(\mathbb{Z})$*. Each* γ *can be represented in the form* $b_1 c w b_2$*; one can solve for the coefficients of* b_1 *and* b_2*. One finds that* $\alpha_1 = \frac{\gamma_{21}}{c_1}$*,* $\alpha_2 = \frac{c_1\gamma_{12} - \gamma_{11}\gamma_{32}}{c_2}$*,* $\alpha_3 = \frac{\gamma_{11}}{c_1}$*,* $\beta_1 = \frac{c_1\gamma_{23} - \gamma_{21}\gamma_{33}}{c_2}$*,* $\beta_2 = \frac{\gamma_{32}}{c_1}$*,* $\beta_3 = \frac{\gamma_{33}}{c_1}$*. Letting* $\psi_M(u) = e^{2\pi i(m_1 u_{1,2} + m_2 u_{2,3})}$ *and analagously for* ψ_N*, we get that*

$$S_w(\psi_M, \psi_N, c) = \sum_{\substack{\gamma_{21}, \gamma_{32} \pmod{c_1} \\ c_1\gamma_{12} - \gamma_{11}\gamma_{32} \pmod{c_2} \\ c_1\gamma_{23} - \gamma_{21}\gamma_{3}3 \pmod{c_3}}} e^{2\pi i\left(m_1 \frac{\gamma_{21}}{c_1} + m_2 \frac{c_1\gamma_{12} - \gamma_{11}\gamma_{12}}{c_2}\right)} e^{2\pi i(n_1 \dots)}.$$

For more details, see Dorian's book.

One can define the **Kloosterman zeta function** for $s = (s_1, \dots, s_{n-1}) \in \mathbb{C}^{n-1}$, which converges for $\mathrm{Re}(s)$ sufficiently large:

$$Z(\psi, \psi', s) = \sum_{c_1=1}^{\infty} \cdots \sum_{c_{n-1}=1}^{\infty} \frac{S_w(\psi, \psi', c)}{c_1^{s_1} \dots c_{n-1}^{s_{n-1}}}.$$

Remark 22.2. *Recall the Selberg conjecture for eigenvalues of Maass forms* $\lambda_j \ge \frac{1}{4}$*. Selberg proved a bound of* $\frac{3}{16}$ *using properties of the Kloosterman zeta function.*

Sarnak-Goldfeld (1983) showed a GL(2) Kloosterman zeta function bound.

$$|Z(\psi, \psi', s)| \ll \frac{|s|^{1/2}}{\mathrm{Re}(s) - \frac{1}{2}}.$$

Remark 22.3. *No bound like this has been shown for general* $GL(n)$ *– it is an open problem.*

22.2 Properties of Kloosterman Sums

Here will briefly discuss some properties of Kloosterman sums.
Fix ψ_M, ψ_N, c, and c. One can show that there exists $\psi_{N'}$ and $\psi_{N''}$ such that

$$S_w(\psi_M, \psi_N, cc') = S_w(\psi_M, \psi_{N'}, c)S_w(\psi_M, \psi_{N''}, c').$$

Theorem 22.4 (Friedberg, 1987)**.** *$S_w(\psi_M, \psi_N, c) \neq 0$ iff w is of the form*

$$\begin{pmatrix} & & & I_{i_1} \\ & & I_{i_2} & \\ & \ddots & & \\ I_{i_\ell} & & & \end{pmatrix},$$

with each I_{i_j} an identity matrix.

22.3 Fourier Expansion of Poincare Series

Recall that we had

$$P^M(g) = \sum_{\gamma \in U_n(\mathbb{Z})\backslash \mathrm{SL}(n,\mathbb{Z})} p(M^*\gamma g)\psi(\gamma g),$$

where $p : \mathfrak{h}^n \to \mathbb{C}$ is a test function with $p(xy) = p(y)$. To compute the geometric side of the KTF, we need to compute the Fourier coefficients of the Poincare series.

Theorem 22.5. *Let $U_w(\mathbb{Z}) = (w^{-1}U_n(\mathbb{Z})w) \cap U_n(\mathbb{Z})$ and $\overline{U_w(\mathbb{Z})} = (w^{-1}U_n(\mathbb{Z})^T w) \cap U_n(\mathbb{Z})$ (and analogously for $\mathbb{R}$). Then*

$$\int_{U_n(\mathbb{Z})\backslash U_n(\mathbb{R})} P^M(ug)\overline{\psi_N(u)}\mathrm{d}^*u = \sum_{w \in W_n} \sum_{c_1=1}^{\infty} \cdots \sum_{c_{n-1}=1}^{\infty} S_w(\psi_M, \psi_N, c)J_w(g, \psi_M, \psi_N, c),$$

where J_w is the Kloosterman integral

$$J_w(g, \psi_M, \psi_N, c) = \left(\int_{\overline{U_w(\mathbb{R})}} p(M^*cwu_2g)\psi_M(wu_2g)\mathrm{d}^*u_2\right)\left(\int_{U_n(\mathbb{Z})\backslash U_w(\mathbb{R})} \psi_M(u_1)\overline{\psi_N(u_1)}\mathrm{d}^*u_1\right).$$

TODO: There might be a mistake with the ψ_M terms – see next lecture for possible correct term.

Proof. One can write

$$\int_{U_n(\mathbb{Z})\backslash U_n(\mathbb{R})} P^M(ug)\overline{\psi_N(u)}\mathrm{d}^*u = \int_{U_n(\mathbb{Z})\backslash U_n(\mathbb{R})} \sum_{\gamma \in U_n(\mathbb{Z})\backslash \mathrm{SL}(n,\mathbb{Z})} p(M^*\gamma ug)\psi_m(\gamma ug)\overline{\psi_N(u)}\mathrm{d}^*u.$$

Recall that $G_w = U_n D_n w U_n$, and our sum over γ can be split into a sum over $U_n(\mathbb{Z})\backslash(\mathrm{SL}(n,\mathbb{Z}) \cap G_w)$ over all w. Letting $\Gamma_w = (w^{-1}U_n(\mathbb{Z})^T w) \cap U_n(\mathbb{Z})$, we can actually rewrite as sum over double coset representatives of $U_n(\mathbb{Z})\backslash(\mathrm{SL}(n,\mathbb{Z}) \cap G_w)/\Gamma_w$. Hence our sum can be rewritten in the form

$$\sum_{w \in W_n} \sum_{c} \sum_{\substack{b_1, b_2 \in U_n(\mathbb{Q}) \\ b_1cwb_2 \in G_w/\Gamma_w}} \sum_{\tau \in \Gamma_w} \int_{U_n(\mathbb{Z})\backslash U_n(\mathbb{R})} p(M^*b_1cwb_2\tau ug)\psi_M(b_1cwb_2\tau ug)\overline{\psi_N(u)}\mathrm{d}^*u.$$

One can show that $p(M^*b_1cwb_2\tau ug) = p(M^*cwb_2\tau ug)$, as $M^*b_1 = b_1'M^*$ for some $b_1 \in U_n(\mathbb{R})$, and using that $p(xy) = p(y)$. Moreover, one can show that

$$\int_{U_n(\mathbb{Z})\backslash U_n(\mathbb{R})} = \int_{U_w(\mathbb{Z})\backslash U_w(\mathbb{R})} \int_{\overline{U_w(\mathbb{Z})}\backslash \overline{U_w(\mathbb{R})}},$$

using that $U_w \cdot \overline{U_w} = U_n$. Using that ψ_M is multiplicative, and a transformation from $u \mapsto \tau^{-1}u$ that sums over all shifts of $\overline{U_w}(\mathbb{Z})\backslash\overline{U_n(\mathbb{R})}$. Hence our integral can be rewritten in the form

$$\sum_{w\in W}\sum_{c}\sum_{\substack{b_1,b_2\in U_n(\mathbb{Q})\\ b_1cb_2\in G_w/\Gamma_w}}\psi_M(b_1)\int_{U_w(\mathbb{Z})\backslash U_w(\mathbb{R})}\int_{\overline{U_w(\mathbb{R})}}p(M^*cwb_2ug)\psi_M(wb_2ug)\overline{\psi_N(u)}\mathrm{d}^*u_2\mathrm{d}^*u_1,$$

where $u = u_1u_2$ with $u_1 \in U_w(\mathbb{Z})\backslash U_w(\mathbb{R})$ and $u_2 \in \overline{U_w(\mathbb{R})}$. Making the transformation $u_1 \mapsto b_2^{-1}u_1$ and splitting the integral, and summing over b_1 and b_2 gives

$$\sum_{w\in W}\sum_{c}S_w(\psi_M,\psi_N,c)\left(\int_{U_w(\mathbb{Z})\backslash U_w(\mathbb{R})}\psi_M(u_1)\overline{\psi_N(u_1)}\mathrm{d}^*u_1\right)\left(\int_{\overline{U_w(\mathbb{R})}}p(M^*cwu_2g)\overline{\psi_N(u_2)}\mathrm{d}^*u_2\right).$$

Here we use the fact that $wu_1 = u_1'w$ for some $u_1' \in U_n(\mathbb{R})$, since $u_1 \in U_w(\mathbb{R})$, to get rid of the u_1 in the p. $\square$

Hence the Nth Fourier coefficient of p^M is of the form

$$\sum_{w}\sum_{c}S_w(\psi_M,\psi_N,c)J_w(g,\psi_M,\psi_n,c).$$

On the geometric side of the Kuznetsov trace formula, we get inner products of the form

$$\left\langle\sum_{M}\sum_{w}\sum_{c}S_wJ_w(g),P^N\right\rangle.$$

Unraveling and summing over w gives Whittaker transforms of the Kloosterman sums. We can approximate these using the trivial bound on Kloosterman sums.

Remark 22.6. *The proof of the trivial bound for $SL(n,\mathbb{Z})$ Kloosterman sums is actually difficult – it was proven by Mark Reader.*

23 Lecture 23 - 4/22/25

I am very busy, so these notes will be unedited and contain more mistakes than usual. I hope to get them edited by the end of May.

23.1 Geometric Side of KTF

We will normalize the Poincare series

$$P^M(g) = b_M \sum_{\gamma \in U_n(\mathbb{Z})\backslash \mathrm{SL}(n,\mathbb{Z})} p(M^*\gamma g)\psi_M(\gamma g),$$

where $b_M = \prod_{k=1}^{n-1} m_k^{-k(n-k)/2}$, and $p : \mathfrak{h}^n \to \mathbb{C}$ and $p(ug) = p(g)$ for all $u \in U_n(\mathbb{R})$, with $M = (m_1, \ldots, m_{n-1})$ and M^* as usual.
Last time, we stated the Fourier expansion:

$$\int_{U_n(\mathbb{Z})\backslash U_n(\mathbb{R})} P^M(ug)\overline{\psi_N(u)}\mathrm{d}^*u = \sum_{w\in W_n}\sum_{c_1=1}^{\infty}\cdots\sum_{c_{n-1}=1}^{\infty} S_w(\psi_M,\psi_N,c)J_w(g,\psi_M,\psi_N,c),$$

where

$$J_w(g,\psi_M,\psi_N,c) = \int_{\overline{U_w(\mathbb{R})}} p(M^*cwu_2g)\psi_M(wu_2g)\overline{\psi_N(u_2)}\mathrm{d}^*u_2 \int_{U_n(\mathbb{Z})\backslash U_n(\mathbb{R})} \psi_M(u_1')\psi_N(u_1)\mathrm{d}^*u_1,$$

where $u_1' = wu_1w^{-1}$ TODO: The last ψ_N should be conjugated? Does that $\psi_M(u_2)$ term exist?
Now on the spectral side of the Kuznetsov trace formula, we get that

$$\left\langle P^M, P^N \right\rangle = C + E$$

where C is the cuspidal part

$$C = \sum_{j=1}^{\infty} A_j(N)\overline{A_j(M)}|P^\sharp(\alpha^{(j)})|^2,$$

where ϕ_j is an orthonormal basis of Maass forms, with $A_j(N)$ the Nth coefficient of ϕ_j and $\alpha^{(j)}$ the Langlands parameter of ϕ_j, and E is the Eisenstein part

$$E = \sum_{P_{n_1,\ldots,n_r}} c_{P_{n_1,\ldots,n_r}} \sum_{\Phi=\phi_1\otimes\cdots\otimes\phi_r} \int_{\substack{n_1s_1+\cdots+n_rs_r=0 \\ \mathrm{Re}(s)=0}} A_{P,\Phi}(N,s)\overline{A_{P,\Phi}(M,s)}|P^\sharp(\alpha_{P,\Phi}(s))|^2,$$

where $A_{P,\Phi}(N,s)$ is the Nth Fourier coefficient of $E_{P,\Phi}$.
Now, for the geometric side, we unravel P^N and use the Fourier expansion of P^M. We will show that

$$\left\langle P^M, P^N \right\rangle = M + K,$$

where M is the main term and K is the Kloosterman term. Here

$$M = b_{MN}\sum_{c_1=1}^{\infty}\cdots\sum_{c_{n-1}=1}^{\infty} S_{I_n}(\psi_M,\psi_N,c)\int_{y_1=0}^{\infty}\cdots\int_{y_{n-1}=0}^{\infty} J_{I_n}(y,\psi_M,\psi_N,c)\overline{p(N^*y)}\prod_{k=1}^{n-1}\frac{\mathrm{d}y_k}{y_k^{k(n-k)+1}},$$

and

$$K = b_{MN}\sum_{w\neq I_n\in W_n}\sum_{c_1=1}^{\infty}\cdots\sum_{c_{n-1}=1}^{\infty} S_s(\psi_M,\psi_N,c)\int_{y_1=0}^{\infty}\cdots\int_{y_{n-1}=0}^{\infty} J_s(y,\psi_M,\psi_N,c)\overline{p(N^*y)}\prod_{k=1}^{n-1}\frac{\mathrm{d}y_k}{y_k^{k(n-k)+1}}$$

.

Proof. We have that

$$\begin{aligned}\langle P^M, P^N\rangle &= b_N \int_{\Gamma_n\backslash\mathfrak{h}^n} P^M(g)\left(\sum_{\gamma\in U_n(\mathbb{Z})\backslash\Gamma_n} \overline{p(N^*\gamma g)\psi_N(\gamma g)}\right)\mathrm{d}^*g \\ &= b_N \int_{U_n(\mathbb{Z})\backslash\mathfrak{h}^n} P^M(g)\overline{p(N^*g)\psi_N(g)}\mathrm{d}^*g \\ &= b_N \int_{U_n(\mathbb{Z})\backslash U_n(\mathbb{R})}\int_{y_1=0}^{\infty}\cdots\int_{y_n}^{\infty} P^M(uy)\overline{p(N^*uy)\psi_N(uy)}\mathrm{d}^*u\mathrm{d}^*y \\ &= b_N \int_{U_n(\mathbb{Z})\backslash U_n(\mathbb{R})}\int_{y_1=0}^{\infty}\cdots\int_{y_n}^{\infty} P^M(uy)\overline{p(N^*y)\psi_N(u)}\mathrm{d}^*u\mathrm{d}^*y,\end{aligned}$$

where we use that $uy = y'u$ for some diagonal matrix y' (and similarly $N^*u = u'N^*$) and ψ is invariant under diagonal matrices. Manipulating more gives

$$\int_{y_1=0}^{\infty}\cdots\int_{y_n}^{\infty} b_N \int_{U_n(\mathbb{Z})\backslash U_n(\mathbb{R})} P^M(uy)\overline{\psi_N(u)}\mathrm{d}^*u\overline{p(N^*y)}\prod_{k=1}^{n-1}\frac{\mathrm{d}y_k}{y_k^{k(n-k)+1}}.$$

Thus, using the Fourier expansion,

$$\langle P^M, P^N\rangle = b_{MN}\sum_{w\in W_n}\sum_{c_1=1}^{\infty}\cdots\sum_{c_n=1}^{\infty} S_w(\psi_M,\psi_N,c)J_w(y,\psi_M,\psi_N,c)\overline{p(N^*y)}\frac{\mathrm{d}y_k}{y_k^{k(n-k)+1}},$$

as desired. □

The Kuznetsov trace formula is precisely

$$C = M + K - E,$$

where $K - E$ is the error term. By choosing the right test function, we can get the orthogonality relation on the LHS, as desired.

23.2 The Main Term of the KTF

One can show from the definition of Kloosterman sums that

$$S_{I_n}(\psi_M,\psi_N,c) = \begin{cases} 1 & c = I_n \\ 0 & \text{otherwise}\end{cases}.$$

Hence the main term reduces to

$$\begin{aligned}M &= b_{MN}\int_{y_1=0}^{\infty}\cdots\int_{y_{n-1}=0}^{\infty} J_{I_n}(y,\psi_M,\psi_N,I_n)\overline{p(N^*y)}\frac{\mathrm{d}y_k}{y_k^{k(n-k)+1}} \\ &= b_{MN}\int_{y_1=0}^{\infty}\cdots\int_{y_{n-1}=0}^{\infty}\left(\int_{\overline{U_{I_n}(\mathbb{R})}} p(M^*u_2y)\psi_M(u_2y)\overline{\psi_N(u_2)}\mathrm{d}^*u_2\int_{U_n(\mathbb{Z})\backslash U_n(\mathbb{R})}\psi_M(u_1)\overline{\psi_N(u_1)}\mathrm{d}^*u_1\right)\overline{p(N^*y)}\frac{\mathrm{d}y_k}{y_k^{k(n-k)+1}}.\end{aligned}$$

Now, note that

$$\int_{U_n(\mathbb{Z})\backslash U_n(\mathbb{R})}\psi_M(u_1)\overline{\psi_N(u_1)}\mathrm{d}^*u_1 = \delta_{M=N},$$

and $\overline{U_{I_n}(\mathbb{R})}$ is the trivial group. Hence,

$$M = \delta_{M=N}b_{MN}\int_{y_1=0}^{\infty}\cdots\int_{y_{n-1}=0}^{\infty} p(M^*y)\overline{p(N^*y)}\frac{\mathrm{d}y_k}{y_k^{k(n-k)+1}} = \delta_{M=N}\int_{y_1=0}^{\infty}\cdots\int_{y_{n-1}=0}^{\infty}|p(y)|^2\mathrm{d}^*y,$$

and by Plancherel, this is equal to

$$M = \delta_{M=N}\int_0^{\infty}\cdots\int_0^{\infty}|P^\sharp(y)|^2\mathrm{d}^*y.$$

The proof of the Plancherel formula is the same for GL(n) as for GL(2).

23.3 Choice of Test Function $P^\sharp$

Recall that $P^\sharp$ is a function from Langlands parameters to $\mathbb{C}$. We want to choose $P^\sharp$ to be a polynomial multiplied by a Gaussian $e^{\frac{\alpha_1^2+\cdots+\alpha_n^2}{T}}$.

Remark 23.1. *Assuming that all the Maass forms are tempered; i.e. the α_i are pure imaginary, this sum of squares is effectively $\alpha_j = ir_j$. It's conjectured that this is true – it's proved for $SL(2,\mathbb{Z})$. For applications, its sufficient to show that the contribution of non-tempered Maass forms is small.*

Our polynomial $F_R(\alpha)$, dependent on a fixed even integer R, will be

$$F_R(\alpha) = \prod_{j=1}^{n-2} \prod_{\substack{K,L\subseteq\{1,2,\ldots,n\} \\ |K|=|L|=j}} \left(1 + \sum_{k\in K}\alpha_k - \sum_{\ell\in L}\alpha_\ell\right)^{R/2}.$$

Remark 23.2. *You can show this for other polynomials, but this polynomial gives better error terms.*

Now, we choose

$$P^\sharp_{R,T}(\alpha) = e^{\frac{\alpha_1^2+\cdots+\alpha_n^2}{2T^2}} F_R\left(\frac{\alpha}{2}\right) \prod_{1\le j\ne k\le n} \Gamma\left(\frac{1+2R+\alpha_j-\alpha_k}{2}\right).$$

One can show that

$$|F_R(\alpha)| \ll T^{R\cdot D(n)+\varepsilon}$$

for $\alpha_1^2+\cdots+\alpha_n^2 \ll T$, where

$$D(n) = \frac{1}{2}\binom{2n}{n} - \frac{n(n-1)}{2} - 2^{n-1}$$

is related to the degree of the polynomial. Combining with Stirling's formula on the Γ functions will give the main term:

Theorem 23.3 (Main term of KTF)**.**

$$M = \delta_{M=N}\sum_{i=1}^{n-1} c_i T^{R(2D(n)+n(n-1))+n-i} + O(T^{R(2D(n)+n(n-1))})$$

With this choice, we can show:

Theorem 23.4. *Let $\lambda_j(M)$ be the Mth Hecke eigenvalue of ϕ_j. Then*

$$\sum_{j=1}^{\infty} \lambda_j(M)\overline{\lambda_j(N)} \frac{|P^\sharp_{T,R}(\alpha)|^2}{\prod_{1\le j\ne k\le n}\Gamma\left(\frac{1+\alpha_j-\alpha_k}{2}\right)} = M + O((MN)^{\frac{n^2+13}{4}} T^{R\binom{2n}{n}-2^n-\varepsilon}).$$

This was proven unconditionally for $n = 2,3,4,5$.

24 Lecture 24 - 4/24/25

I am very busy, so these notes will be unedited and contain more mistakes than usual. I hope to get them edited by the end of May.
I am also not here for the last two lectures – so the notes for those will be from someone else.

24.1 Rankin-Selberg Convolutions

There are three cases to consider:

- $\mathrm{GL}(n) \times \mathrm{GL}(n)$: Done in 1940 for $\mathrm{GL}(2) \times \mathrm{GL}(2)$ by Rankin and Selberg. Done for general n by Jacquet and Piatetski-Shapiro. This case involves Eisenstein series.
- $\mathrm{GL}(n) \times \mathrm{GL}(n+1)$: The easiest case.
- $\mathrm{GL}(n) \times \mathrm{GL}(m)$, with $n < m-1$: Also done by Jacquet for Piatetski-Shapiro for special cases. Done in general by Cogdell and Piatetski-Shapiro (done adelically).

The $\mathrm{GL}(n) \times \mathrm{GL}(n)$ case involves Eisenstein series. If one integrates

$$\int_{\mathrm{SL}(n,\mathbb{Z})\backslash\mathfrak{h}^n} \phi(g)\eta(g)E_{B_n}(g,s)\mathrm{d}^*g$$

by unraveling E_{B_n} and expanding the Fourier expansion for η and unravel the $U(n-1,\mathbb{Z})\backslash\mathrm{SL}(n-1,\mathbb{Z})$ sum in the Fourier expansion for η, one gets that

$$L(s,\phi\times\eta) = \zeta(ns)\sum_{m_1=1}^{\infty}\cdots\sum_{m_n=1}^{\infty}\frac{A(M)B(M)}{(m_1^{n-1}m_2^{n-2}\dots m_{n-1})^s},$$

where $M = (m_1,\dots,m_{n-1})$. Using the functional equation/analytic continutation for the Eisenstein series, you can show the same for L. (We did this previously.)

24.2 Rankin-Selberg for $\mathbf{GL}(n) \times \mathbf{GL}(n+1)$

Let ϕ be a Maass form for $\mathrm{SL}(n,\mathbb{Z})$, and let η be a Maass form for $\mathrm{SL}(n+1,\mathbb{Z})$. Let M and M^* be defined as previously, and let $b_M = \prod_{k=1}^{n-1} m_k^{\frac{k(n-k)}{2}}$. Then recall that we have the Fourier expansion

$$\phi(g) = \sum_{\gamma\in U_{n-1}(\mathbb{Z})\backslash\mathrm{SL}(n-1,\mathbb{Z})}\sum_{m_1=1}^{\infty}\cdots\sum_{m_{n-2}=1}^{\infty}\sum_{m_{n-1}\neq 0}\frac{A(M)}{b_M}W_{\alpha_\phi}^{\mathrm{sign}(m_{n-1})}\left(M^*\begin{pmatrix}\gamma & \\ & 1\end{pmatrix}g\right)$$

and similarly

$$\phi\left(\begin{pmatrix}g & \\ & 1\end{pmatrix}\right) = \sum_{\gamma\in U_n(\mathbb{Z})\backslash\mathrm{SL}(n,\mathbb{Z})}\sum_{m_1=1}^{\infty}\cdots\sum_{m_{n-1}=1}^{\infty}\sum_{m_n\neq 0}\frac{B(M)}{b_M}W_{\alpha_\eta}^{\mathrm{sign}(m_n)}\left(M^*\begin{pmatrix}\gamma & \\ & 1\end{pmatrix}\begin{pmatrix}g & \\ & 1\end{pmatrix}\right).$$

Here's we'll unravel the sum in the Fourier expansion for η – there is no Eisenstein series here! In particular, we compute

$$\begin{aligned}\left\langle\phi,\overline{\eta}\det(*)^{\overline{s}-1/2}\right\rangle &= \int_{\mathrm{SL}(n,\mathbb{Z})\backslash\mathfrak{h}^n}\phi(g)\overline{\eta\left(\begin{pmatrix}g & \\ & 1\end{pmatrix}\right)}\det(g)^{s-1/2}\mathrm{d}^*g \\ &= \sum_{\gamma\in U_n(\mathbb{Z})\backslash\mathrm{SL}(n,\mathbb{Z})}\sum_M\frac{B(m_1,\dots,m_n)}{\prod_{k=1}^n m_k^{\frac{k(n-k)}{2}}}\int_{\mathrm{SL}(n,\mathbb{Z})\backslash\mathfrak{h}^n}\phi(g)W_{\alpha_\eta}^{\mathrm{sign}(m_n)}\left(M^*\begin{pmatrix}\gamma & \\ & 1\end{pmatrix}\begin{pmatrix}g & \\ & 1\end{pmatrix}\right)\det(g)^{s-1/2}\mathrm{d}^*g\end{aligned}$$

Making the change of variable $g \to \gamma^{-1} g$ gives

$$\left\langle \phi, \overline{\eta} \det(*)^{\overline{s}-1/2} \right\rangle = \sum_M \frac{B(m_1, \ldots, m_n)}{\prod_{k=1}^n m_k^{\frac{k(n-k)}{2}}} \sum_{\gamma \in U_n(\mathbb{Z}) \backslash \mathrm{SL}(n,\mathbb{Z})} \int_{\gamma^{-1}(\mathrm{SL}(n,\mathbb{Z}) \backslash \mathfrak{h}^n)} \phi(g) W_{\alpha_\eta}^{\mathrm{sign}(m_n)} \left(M^* \begin{pmatrix} g & \\ & 1 \end{pmatrix} \right) \det(g)^{s-1/2} \mathrm{d}^* g$$

$$= \int_{U_n(\mathbb{Z}) \backslash \mathfrak{h}^n} \sum_M \frac{B(m_1, \ldots, m_n)}{\prod_{k=1}^n m_k^{\frac{k(n-k)}{2}}} \phi(g) W_{\alpha_\eta}^{\mathrm{sign}(m_n)} \left(M^* \begin{pmatrix} g & \\ & 1 \end{pmatrix} \right) \det(g)^{s-1/2} \mathrm{d}^* g$$

$$= \sum_M \int_{U_n(\mathbb{Z}) \backslash U_n(\mathbb{R})} \frac{B(m_1, \ldots, m_n)}{\prod_{k=1}^n m_k^{\frac{k(n-k)}{2}}} \int_{y_1=0}^\infty \cdots \int_{y_{n-1}=0}^\infty \phi(xy) W_{\alpha_\eta}^{\mathrm{sign}(m_n)} \left(M^* \begin{pmatrix} xy & \\ & 1 \end{pmatrix} \right) \det(y)^{s-1/2} \mathrm{d}^* y \mathrm{d}^* x.$$

We make the transformation $y_1 \to y_1/m_1$. Note that the integral over x picks off the $(m_2, \ldots, m_{n-1})$th Fourier coefficient of ϕ. We can then unravel the Fourier decomposition of ϕ. Finally, we continue to make the transformations $y_i \to y_i/m_{i+1}$. In the end, we get

$$\sum_{m_1=1}^\infty \cdots \sum_{m_{n-1}=1}^\infty \sum_{m_n \neq 0} \frac{B(m_1, \ldots, m_n)}{b_M} \int_{U_n(\mathbb{Z}) \backslash \mathfrak{h}^n} \phi(g) W_{\alpha_\eta}^{\mathrm{sign}(m_n)} \left(\left(\begin{matrix} m_2 \ldots |m_n| y_1 \ldots y_{n-1} & & & \\ & m_2 \ldots m_{n-1} y_1 \ldots y_{n-2} & & \\ & & \ddots & \\ & & & m_2 y_1 \end{matrix} \right.\right.$$

$$= \sum_{m_1=1}^\infty \cdots \sum_{m_{n-1}=1}^\infty \sum_{m_n \neq 0} \frac{A(m_2, \ldots, m_n) B(m_1, \ldots, m_n)}{(m_1^n m_2^{n-1} \ldots |m_n|)^s} \int_{y_1=0}^\infty \cdots \int_{y_{n-1}=0}^\infty W_{\alpha_\phi}(y) \overline{W_{\alpha_\eta} \begin{pmatrix} y & \\ & 1 \end{pmatrix}} (\det y)^{s-1/2} \prod_{k=1}^{n-1} y_k^{-k(n-k)} \, \mathrm{d}y_k / y_k$$

This is some sort of generalized Mellin transform; note that $L(s, \phi \times \eta)$ is the left term in the product. Since there is no Eisenstein series, we need to prove the functional equation in a direct way. Define

$$\Lambda(s, \phi \times \overline{\eta}) := \left\langle \phi \overline{\eta}, \det(*)^{\overline{s}-1/2} \right\rangle.$$

Theorem 24.1. *We have the functional equation*

$$\Lambda(s, \phi \times \overline{\eta}) = \Lambda(1-s, \widetilde{\phi} \times \overline{\widetilde{\eta}}),$$

where here $\widetilde{\phi} = \phi(w_n (g^{-1})^T w_n)$ is the dual Maass form (associated to the contragradient representation), with w_n the long Weyl element (all 1s on the antidiagonal).

Proof. Let $g^\iota = w_n (g^{-1})^T w_n$. Then note that g^ι has Iwasawa decomposition

$$g^\iota = \begin{pmatrix} 1 & (-1)^{\lfloor n/2 \rfloor + 1} & & & \\ & 1 & -x_2 & & \\ & & \ddots & \ddots & \\ & & & 1 & -x_{n-1} \\ & & & & 1 \end{pmatrix} \begin{pmatrix} y_1 \cdots y_{n-1} & & & & \\ & y_2 \cdots y_{n-1} & & & \\ & & \ddots & & \\ & & & y_{n-1} & \\ & & & & 1 \end{pmatrix}.$$

Then

$$\Lambda(s, \phi \times \eta) = \left\langle \phi \eta, \det(*)^{\overline{s}-1/2} \right\rangle$$
$$= \int_{\mathrm{SL}(n,\mathbb{Z}) \backslash \mathfrak{h}^n} \phi(g) \eta \left(\begin{pmatrix} g & \\ & 1 \end{pmatrix} \right) (\det g)^{s-1/2} \mathrm{d}^* g$$
$$= \int_{\mathrm{SL}(n,\mathbb{Z}) \backslash \mathfrak{h}^n} \phi(g^\iota) \eta \left(\begin{pmatrix} g^\iota & \\ & 1 \end{pmatrix} \right) (\det g)^{s-1/2} \mathrm{d}^* g.$$

Note that $\mathrm{d}^* g$ is invariant under $g \to g^\iota$, and $\det(g^\iota)^{s-1/2} = \det(g)^{1/2-s}$. Using this gives

$$\Lambda(1-s, \widetilde{\phi} \times \widetilde{\eta}),$$

as desired. □

Remark 24.2. *One can generalize these techniques to unitary groups (no Eisenstein series) - this is called the doubling method.*

24.3 Rankin-Selberg for $\mathbf{GL}(n) \times \mathbf{GL}(n')$

This is the most difficult case – we'll start it today. Here $n < n' - 1$; i.e. $n \leq n'$. We'll need the projection operator – first discovered by Jacquet and Piatestski-Shapiro.

Definition 24.3 (Projection Operator)**.** *Fix $2 \leq n < n' - 1$. The* ***projection operator*** *$P_n^{n'}$ sending Maass forms of $SL(n', \mathbb{Z})$, projecting/mapping them to automorphic forms on the mirabolic subgroup $P_{n,1} \subseteq GL(n+1, \mathbb{R})$.*
Let $g \in P_{n,1}(\mathbb{R})$. We define

$$P_n^{n'}\phi(g) := |\det(g)|^{-\frac{n'-n-1}{2}} \int_0^1 \cdots \int_0^1 \phi \begin{pmatrix} g & & u_{1,n+2} & \cdots & u_{1,n'} \\ & & \vdots & & \vdots \\ & & u_{n+1,n+2} & \cdots & \vdots \\ & & 1 & & \ddots \\ & & & \ddots & u_{n'-1,n'} \\ & & & & 1 \end{pmatrix} e^{-2\pi i(u_{n+1,n+2}+\cdots+u_{n'-1,n'})} \prod_{\substack{n+2\leq j\leq n' \\ 1\leq i<j}} \mathrm{d}u_{i,j}\,.$$

Example 24.4. *Let $n = 2$ and $n' = 4$. Consider $P_{2,1} \subseteq GL(3, \mathbb{R})$, let ϕ be a Maass form for $SL(4, \mathbb{Z})$, and let $g = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix} \subseteq P_{2,1}(\mathbb{R})$. Then*

$$P_2^4(\phi(g)) = \int_0^1 \int_0^1 \int_0^1 \phi \begin{pmatrix} a & b & 0 & u_{1,4} \\ c & d & 0 & u_{2,4} \\ 0 & 0 & 1 & u_{3,4} \\ 0 & 0 & 0 & 1 \end{pmatrix} e^{-2\pi i u_{3,4}} \,\mathrm{d}u_{1,4}\,\mathrm{d}u_{2,4}\,\mathrm{d}u_{3,4}\,.$$

One can evaluate this by expanding the Fourier expansion for ϕ. Then

$$P_2^4\phi(g) = \sum_{\gamma \in U_2(\mathbb{Z})\backslash SL(2,\mathbb{Z})} \sum_{m_2=1}^{\infty} \sum_{m_3 \neq 0} \frac{A_\phi(1, m_2, m_3)}{m_2^2 m_3} W_{\alpha_\phi}^{sign(m_3)} \left(M^* \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} \begin{pmatrix} g & \\ & 1 \end{pmatrix} \right).$$

Next time, we'll go through more examples, and then work through $\mathrm{GL}(n) \times \mathrm{GL}(n)$ using this method.