

Subconvexity Seminar

$$f(z) = \sum_{m=1}^{\infty} b_m m^{k-1/2} q^m$$

Fourier expansion for ^(Hecke) cusp form of weight k .

Want ^(arithmetic) q -aspect subconvexity for twisted L-fns

$$L_f(s, \chi) = \sum_m b_m \chi(m) m^{-s}$$

χ_0 primitive of conductor q .

Fn eq: $\left(\frac{q}{2\pi}\right)^s \frac{\Gamma(s + \frac{1}{2}(k+1))}{\Gamma(\frac{s + \frac{1}{2}(k+1)}{2}) \Gamma(\frac{s + \frac{1}{2}(k+1)}{2})} L_f(s, \chi) = \epsilon_{\chi}^2 i^k \left(\frac{q}{2\pi}\right)^{1-s} \frac{\Gamma(1-s + \frac{1}{2}(k-1))}{\Gamma(\frac{1-s + \frac{1}{2}(k-1)}{2}) \Gamma(\frac{1-s + \frac{1}{2}(k-1)}{2})} L_f(1-s, \bar{\chi})$

$\epsilon_{\chi} = \tau(\chi) q^{-1/2}$ note that $|\epsilon_{\chi}| = 1$

$\tau(\chi) = \sum_{a \pmod{q}} \chi(a) e\left(\frac{a}{q}\right)$ Gauss sum

Convexity bound gives

$$L_f(s, \chi) \ll q^{1/2} (\log q)^2 (q^{1/2} + o(1))$$

Main Thm: (Duke - Friedlander - Iwaniec):

$$L_f(s, \chi_0) \ll |s|^2 q^{5/11} \underbrace{\tau(q)^2}_{\# \text{ of divisors}} \log q$$

all the non $q^{5/11}$ terms are super unoptimized.

Main Ideas:

1. Approximate FE

- coming from Voronoi summation

2. Delta method (version of circle method)

- write $\delta(n)$ as exponential sum

- (2)
- "separate structure into two things twisted by $GL(1)$, giving new FEs of lower conductor i.e. length/complexity"

3. Amplification method

- We compute a bound over all primitive χ but want to isolate to a specific χ_0
- said function will make χ_0 contribution larger.
- Decrease size of main term and increase size of error to gain a little

Start: Let $s = \frac{1}{2} + it$. want a bound for

B.3 idea 1

$$L_f(s, \chi_0) = \sum_m b_m \chi_0(m) m^{-s}$$

dyadically; i.e.

Start by estimating sums of type

$$H = \sum_m b_m \chi_0(m) m^{-s} h(m)$$

where $h(m)$ is ^{smooth} ~~compactly~~ supported on $[M, 2M]$ and has sufficiently small growth ($h^{(j)}(m) \ll M^{-j}$).

Can only go so high. Cutoff at M and use Voronoi summation to get dual sum.

Voronoi summation: $c \geq 1$, $\gcd(a, c) = 1$, F smooth, cpt support on $\mathbb{R}^+$

$$\sum_m b_m e\left(\frac{am}{c}\right) F(m) = \sum b_r e\left(-\frac{\bar{a}r}{c}\right) \check{F}(r)$$

where $a\bar{a} \equiv 1 \pmod{c}$ and $\check{F}(r)$ is Hankel transform.

$$\check{F}(y) = 2\pi i^k c^{-1} \int_0^\infty F(x) J_{k-1}\left(\frac{4\pi}{c} \sqrt{xy}\right) dx.$$

Rmks: J is usual Bessel fn.

(3)

$$J_\nu(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m+\nu+1)} \left(\frac{x}{2}\right)^{2m+\nu}$$

has asymptotic limiting behavior

$$J_\nu(x) \ll (1+x)^{-1/2}$$

i.e. if $x \gg 1$ $J_\nu(x)$ decays quickly.

modularity of f needed here (used in proof)
(basically only point it is used).

Cor: $\sum_m b_m \chi(m) F(m) = E_x^2 \sum_r b_r \bar{\chi}_0(r) \tilde{F}(r)$

Follows from Voronoi + Gauss sum properties. (see later). $\square$

Hence we can take for m large, dualize and consider dual sum.

One can check that for $F(m) = m^{-s} h(m)$, if h supported on

then $\tilde{F}(r)$ rapidly decays for $r \gg \frac{q^2}{M}$. $(J_{k+1}(\frac{4\pi}{q} \sqrt{M} \frac{r}{q}))$

i.e. dual length sum is $\frac{q^2}{M}$.

$r \gg \frac{q^2}{M}$
means decay.

$$\sum b_m \chi(m) m^{-s} g(m) \quad \text{dual} \quad \sum b_m \chi(m) m^{-s} g(m)$$

$[M, 2M]$
or length
 M

cutoff

$[M, 2M]$
length
 $\frac{q^2}{M}$

will need to balance i.e. choose right cutoff such that errors are same order.

How to get bounds on $\sum b_m \chi(m) F(m)$? Roughly we'll do

$$\left| \sum b_m \chi_0(m) F(m) \right|^2 \leq \sum_{\chi(m) \neq g}^* \left| \sum_m b_m \chi(m) g(m) \right|^2 \quad (\text{amplifier})$$

object next of study

Biz Idea
2

Delta method

Want bounds on something like $n \leq \frac{N}{2}$

$$S = \sum_{\chi \pmod{q}}^* \left| \sum_n a_n \chi(n) \right|^2$$

← sums from before + amplifier will be put in here

Gauss sums + orthogonality

$$\chi(n) = \frac{1}{\tau(\chi)} \sum_{a \pmod{q}} \bar{\chi}(a) e\left(\frac{an}{q}\right).$$

$$S \leq \frac{\varphi(q)}{q} \sum_{a \pmod{q}} \left| \sum_n a_n e\left(\frac{an}{q}\right) \right|^2$$

Swap order of summation gives + (orthogonality)

$$S \leq \varphi(q) \sum_{h \equiv 0 \pmod{q}} S_h$$

$$w/ S_h = \sum_{n_1 - n_2 = h} a_{n_1} \bar{a}_{n_2}$$

$$\delta(n) = \begin{cases} 1 & n=0 \\ 0 & \text{otherwise} \end{cases}$$

For $h \equiv 0 \rightarrow \varphi(q) \sum |a_n|^2$

for $h \not\equiv 0$ effectively a $\sum_{n_1, n_2} \delta(n_1 - n_2 - h) a_{n_1} \bar{a}_{n_2}$ sum.

hope adding exponential sums will help w/ errors.

Delta method:

$w(t)$: even on $\mathbb{R}$, $w(0) = 0$, compact support;

$$\sum_{k=1}^{\infty} w(k) = 1.$$

$$\delta_k(n) := w(k) - w\left(\frac{n}{k}\right).$$

reexpress divisibility condition.

$$\text{Then } \delta(n) = \sum_{k|n} \delta_k(n) = \sum_k \overline{k \sum_{h \pmod{k}} e\left(\frac{hn}{k}\right) \delta_k(n)}$$

$$\Delta_c(n) := \sum_r r^{-1} \delta_{cr}(n)$$

then rewrite

(5)

$$\boxed{\delta(n) = \sum_c c^{-1} \sum_{a \pmod c}^* e\left(\frac{an}{c}\right) \Delta_c(n).}$$

delta method.

• Conditions:

• $w(t)$ supported on $\frac{K}{2} \leq t \leq K$, with $w^{(j)}(t) \ll K^{-j-1}$

• summation up to $\ln K \frac{N}{2}$, with $K = N^{1/2}$.

• Properties:

$\Delta_c(n) \ll K^{-1}$, is 0 if $c > K$. Hence only need to sum c up to K .

Importing back

$$S_h = \sum_{n_1, n_2} \delta(n_1 - n_2 - h) a_{n_1} \overline{a_{n_2}}$$

$$= \sum_c c^{-1} \sum_{a \pmod c}^* \sum_{n_1, n_2} a_{n_1} \overline{a_{n_2}} e\left(\frac{a}{c} (n_1 - n_2 - h)\right) \Delta_c(n_1 - n_2 - h)$$

$$=: S_{hc}.$$

Bj Idea
3

$$\sum_{\chi \pmod q}^* \left| \sum_m b_m \chi(m) g(m) \right|^2 \underbrace{\left| \sum_\ell \lambda_\ell \chi(\ell) \right|^2}_{\text{amplifier.}}$$

λ_ℓ complex numbers from 1 to L .

will eventually let $\lambda_\ell = \overline{\chi_0}(\ell)$. Hence contribution on RHS is big when $\chi = \chi_0$ and small if $\chi \neq \chi_0$ (Heuristic via GRH).
 $\gg L$ $\ll L^{1/2}$

i.e. the a_n we want to consider

$$a_n = \sum_{\ell m = n} \lambda_\ell b_m g(m).$$

$g \in C^2$, support in $[M, 2M]$, with sufficient decay (as well as derivatives)
 $(|g^{(j)}(M)| \leq M^{-j})$.
 $j = 0, 1, 2$

⑥

Specializing S_{hc} to our previous case gives

$$S_{hc} = \sum_{l_1, l_2} \lambda_{l_1} \overline{\lambda_{l_2}} \sum_{a \pmod{c}}^* e\left(\frac{ah}{c}\right) \sum_{m_1, m_2} b_{m_1} \overline{b_{m_2}} e\left(\frac{a}{c}(l_1 m_1 - l_2 m_2)\right) F(m_1, m_2)$$

$T_{l_1, l_2}(c).$

with $F(m_1, m_2) = g(m_1) \overline{g(m_2)} \Delta_c(l_1 m_1 - l_2 m_2 - h).$

With the choice of parameters,

$$K = N^{1/2} = 2(LM)^{1/2}$$

the choices made for g and w (support + growth) mean F has good growth / derivatives good growth.

$$F(i, j) \ll K^{-1} (CM/K)^{-\epsilon_j}.$$

and

We want to rewrite $T_{l_1, l_2}(c)$ in terms of a Kloosterman sum, so we need to dualize again (need an $\bar{a}$) via Voronoi:

$$T_{l_1, l_2}(c) = \sum_{a \pmod{c}}^* e\left(\frac{ah}{c}\right) \sum_{r_1, r_2} b_{r_1} \overline{b_{r_2}} e\left(\frac{\overline{al'}}{c_1} r_1 - \frac{\overline{al''}}{c_2} r_2\right) \check{F}(r_1, r_2)$$

$$l' = \frac{l_1}{(l_1, c)} \quad l'' = \frac{l_2}{(l_2, c)} \quad c_i = \frac{c}{(l_i, c)} \quad (\text{decreasing})$$

F having good growth means $\check{F}$ does. In particular one can show

$$\sum_{r_1, r_2} b_{r_1} \overline{b_{r_2}} |\check{F}(r_1, r_2)| \ll K.$$

$\sum_{r \in X} |b_r|^2 \ll X$
by Parseval

Then

$$T_{l_1, l_2}(c) = \sum_{r_1, r_2} b_{r_1} \overline{b_{r_2}} \check{F}(r_1, r_2) \sum_{a \pmod{c}}^* e\left(\frac{ah \dots + \bar{a} \dots}{c}\right)$$

$$\ll K(h, c)^{1/2} \tau(c) c^{1/2} \\ = (LM)^{1/2} (h, c)^{1/2} \tau(c) c^{1/2}$$

(7)

Weil bound:

$$S(m, n; c) \leq (m, n, c)^{1/2} \tau(c) c^{1/2} \\ \sum_{a \pmod{c}}^* e\left(\frac{am + \bar{a}n}{c}\right)$$

$$S_{hc} \ll (h, c)^{1/2} c^{1/2} \tau(c) (LM)^{1/2} \left(\sum_l |\lambda_l|\right)^2$$

Recall we only need to sum c up to $K = 2(LM)^{1/2}$. then

$$S_h = \sum_c c^{-1} S_{hc} \ll \tau(h) (LM)^{3/4} (\log LM) \left(\sum_l |\lambda_l|\right)^2 \quad \left(\begin{array}{l} \text{use that} \\ \sum_{n \leq x} \tau(n) \sim x \log x \end{array} \right)$$

Summing over all h up to $N = 4LM$ (b/c max difference is N) gives

$$\sum_{\substack{h \equiv 0 \pmod{q} \\ h \neq 0}} S_h \ll \tau(q) (LM)^{7/4} (\log LM)^2 \left(\sum_l |\lambda_l|\right)^2.$$

Moreover

$$S_0 = \varphi(q) \sum_n |a_n|^2 \ll \frac{M(\log M) \sum_l |\lambda_l|^2 \tau(l)}{1} +$$

Thus we have upper bound for

$$\sum_{\chi \pmod{q}}^* \left| \sum_m b_m \chi(m) g(m) \right|^2 \left| \sum_l \lambda_l \chi(l) \right|^2 \ll$$

What happens when $\lambda_l = \bar{\chi}_0(l)$? Then

$$\left| \sum_m b_m \chi_0(m) g(m) \right|^2 \left| \sum_{\substack{(l, q)=1 \\ l \leq L}} 1 \right|^2 \leq M(\log M) \left(\sum_{\substack{(l, q)=1 \\ l \leq L}} \tau(l) \right) \sim L \log L$$

$\swarrow$ lowers main term

$\swarrow$ boosts error.

$$+ \tau(q) (LM)^{7/4} (\log LM)^2 \left(\sum_{\substack{(l, q)=1 \\ l \leq L}} 1 \right)^2$$

Choose L such that errors on RHS are equal

$$L = q^{4/11} M^{-3/11} + 2q \tau(q) \varphi(q)^{-1} \Rightarrow \text{upper bound of} \\ (q^{7/22} M^{7/11} + M^{7/8}) \tau(q)^2 \log M$$

How to finish?

Recall we had normal sums

$$\sum_m b_m \chi(m) \frac{m^{-s} h(m)}{g(m)}$$

basically, but
add an extra
 $M^{-\frac{1}{2}}$ from

m^{-s} .

on $[M, 2M]$, get bound

$$|S|^2 (q^{7/22} M^{3/22} + M^{3/8}) \tau(q)^2 \log M$$

when are
the bounds
equal?

when $M = q$!

Sum over dyadic intervals (but error most concentrated
around $[M, 2M] \approx q$.)
Hence error is $|S|^2 q^{5/11} \tau(q)^2 \log q$.

and dual sums

$$\sum_r b_r \tilde{\chi}(r) \tilde{f}(r)$$

Hankel transform of
 $m^{-s} g(m)$.

Can show

$$\tilde{f}(r) \ll |S|^2 M^{1/2} q^{-1} (1 + q^{-2} M r)^{-5/4}$$

$\Downarrow$

$$|S|^2 (q^{7/22} R^{7/11} + R^{7/8})$$

sum up to
 R .

$$M^{1/2} q^{-1} (1 + q^{-2} M R)^{-5/4} \tau(q)^2 \log R$$

since decay when $R \gg q^2/M$,
take $R = q^2/M$.

□