Subconvexity via Automorphic Periods

- Papers: Bernstein–Reznikov, Michel–Venkatesh, Nelson's works

Period formulas take the form:

$$\int_{[H]} f(h) \qquad f \in L^2(G(\mathbb{Q}) \backslash G(\mathbb{A}))$$

where $H \subseteq G$ a subgroup of a Lie group G.

$$\boxed{\begin{array}{c} Z(s, \phi) = \int_{\mathbb{A}^\times} W_\phi\begin{pmatrix} y & \\ & 1 \end{pmatrix} |y|^{s-\frac{1}{2}} d^\times y \\ \Rightarrow \text{Rankin–Selberg: } GL_2 = H,\ G = GL_2 \times GL_2 \end{array}}$$

Ichino–Watson:

$$\left| \int_{\Gamma \backslash SL_2(\mathbb{R})} v_1(g) v_2(g) v_3(g)\, dg \right|^2 = \frac{L(\frac{1}{2}, \pi_1 \otimes \pi_2 \otimes \pi_3)}{\prod L(1, \mathrm{Ad}\, \pi_i)} \times \underbrace{\text{matrix coeff}}_{L_\infty}$$

where $\|v_i\| = 1$.

Rmk: If v_i's are Maass forms, L_∞ is a product of Gamma factors

$$\Rightarrow L(\tfrac{1}{2}, \otimes \pi_i) \ll_{\pi_1, \pi_2} \text{Period} \cdot \exp(\tfrac{\pi}{2}, |\lambda_3|)$$

So Maass forms give bad bounds.

$\boxed{\text{Goal: What are good vectors?}}$

Michel–Venkatesh uses softer methods than Bernstein–Reznikov to get convexity. Then, adding amplification gave subconvexity. Nelson's work uses the orbit method, pushed results past $GL(3)$.

Bernstein–Reznikov: $\pi \rightsquigarrow \underbrace{\text{Rep theory}}_{\text{simple}} + \underbrace{\text{Arithmetic}}_{L(\frac{1}{2}, \pi)}$

$\ell : V \otimes V' \otimes W \to \mathbb{C}$ trilinear form of GL_2-irreps

Thm: ℓ is unique up to constant

ℓ induces Hermitian form on $E = V \otimes V'$.

$\left(L^2(X) = \hat{\otimes} V_i,\right.$ the discrete spectrum only$\left.\right)$

$\ell_i^{aut} : E \otimes V_i \to \mathbb{C}$ given by $\int \phi \phi' \phi_i$. (gives H_i^{aut})

$\sum H_i^{aut} \leq H_\Delta$, a Hermitian form on $C^\infty(X \times X)$.

Then, construct H_i^{mod}, which by Thm is $a_i \cdot H_i^{aut}$. The a_i give info about $L(\frac{1}{2}, \otimes \pi_i)$.

Choose test vectors on E s.t. $H_i^{mod}(w) \approx \begin{cases} 1 & \text{near } V_i = \pi_0 \\ 0 & \text{else} \end{cases}$

$$\Rightarrow \sum a_i H_i^{mod}(w) \leq H_\Delta(w)$$
$$\text{\rotatebox{90}{$\approx$}}$$
$$|a_{\pi_0}|$$
$$\text{\rotatebox{90}{$\approx$}}$$
$$|L(\tfrac{1}{2}, \otimes \pi_i)|.$$

Also, ~~Nelson~~ Bernstein-Reznikov showed $H_i^{mod}(w) \approx 1$ on π such that $|\mu_i| \in [T, 2T]$.

$$\sum L(\tfrac{1}{2}, \otimes \pi_i) \leq H_\Delta(w).$$

Michel-Venkatesh

$H \subseteq G$, $P(\phi, \varphi) = \int_{[H]} \underset{\substack{\downarrow \\ \text{of } G \\ \cap \\ \pi}}{\phi(h)} \; \underset{\substack{\downarrow \\ \text{aut form} \\ \text{of } H!}}{\varphi(h)} \, dh$ $\subset \sigma$

For so-called GGP pairs

$$\left| \int_{[H]} \phi(h) \varphi(h) \, dh \right|^2 = c(\pi, \sigma) \Lambda(\pi \times \sigma, \tfrac{1}{2}) \prod_v \int_{H(\mathbb{Q}_v)} \frac{\langle h\phi_v, \phi_v \rangle \langle h\varphi_v, \varphi_v \rangle}{\|\phi_v\|^2 \|\varphi_v\|^2} \, dh.$$

Jacquet-Langlands did this for $GL_2 \times GL_1$.

$$\sum_{\text{avg}} \frac{P(\phi, \varphi)}{(\|\phi\| \|\varphi\|)^2}$$ vary φ "vertical", ϕ "horizontal"

Vertical: $\sum |P(\phi, \varphi)|^2 = \int_{[H]} |\phi(h)|^2 \, dh$ (Plancherel)

$A = GL_1 \hookrightarrow GL_2$

$x \mapsto \begin{pmatrix} x & \\ & 1 \end{pmatrix}$

J-L

$$\sum (L(\pi \times \chi, \tfrac{1}{2}))^2 \times \text{matrix coeff}$$

+ amplification

$\Rightarrow$ subconvexity.

<u>Relative Trace Formula:</u>

$f \in C_c^\infty(G(\mathbb{A}))$, $\phi \mapsto R(f)\phi$, kernel : $K_f(x, y)$

$$\sum_{\pi \in A(h)} \sum_{\phi \in \mathcal{B}(\pi)} R_f(\phi)(x) \overline{\phi(y)} = \sum_{\gamma \in G(\mathbb{Q})} f(x^{-1} \gamma y)$$

Integrate both sides over $H \times H$. By Jacquet-Langlands, get $L(\tfrac{1}{2}, \pi \times \sigma) W_f(\phi, \varphi) \times$ matrix coeff. <u>Pick f to restrict to desired family</u> (Goal)

Talk 2 — Orbit ~~method~~ & Sub~~convexity~~ seminar.

~~Orbit method & SGP~~ **Microlocal analysis**

Maass forms ϕ_i, $\int_{\Gamma\backslash H} \phi_1 \phi_2 \phi_3$ = Hard to analyse.
(Good : difficult hypergeom fns)

\# Basic problem (Toy model) : $L^2(S^2) = \bigoplus_{\ell \in \mathbb{N}} V_\ell$ $\dim V_\ell = 2\ell+1$

(Spherical harmonics irrep of $SO(3)$)

V_ℓ = Harmonic ($\Delta f = 0$) poly hom. in x,y,z deg ℓ.

$V_\ell = \bigoplus_m Y_{\ell,m}$ st $Y_{\ell,\ell} = (x+iy)^\ell$

(double cover of $SU(2)$)

$$\int_{S^2} Y_{\ell_1,m_1} Y_{\ell_2,m_2} Y_{\ell_3 m_3} = \boxed{?}$$

Very hard problem. $G = SO(3)$, $\mathfrak{g} = \mathbb{R}^3$, $\hat{\mathfrak{g}} = \mathbb{R}^3$

Step 1) Idea: Chop up $f = \sum_{\alpha \in \mathbb{R}^n \oplus \mathbb{R}^n} f_\alpha$ (position sp / freq sp) f_α localised in space & frequency.
f_α, $\hat{f}_\alpha$ reasonable behavior
(can't be completely loc due to Heisenberg uncertainty)

Step 2) Decompose $Y_{\ell_i, m_i} = \int c_\alpha v_\alpha$ ← Microlocalised at α

$$\int_{S^2} Y_{\ell_1,m_1} \cdots Y_{\ell_3,m_3} = \int_{\alpha,\beta,\gamma} \cdots \int_{S^2} v_\alpha v_\beta v_\gamma$$

Much easier to evaluate.
$= 0$
unless $\ell_1\alpha + \ell_2\beta + \ell_3\gamma = 0$.

$GL_{n+1}\times GL_n$, $U_{n+1}\times U_n$

Case of SCP $\quad$ π, σ GGP pairs: $[H] = G(\mathbb{Q})\backslash G(\mathbb{A})$

GGP $\Rightarrow$ $\left|\int_{[H]} \Phi(y)\,\phi(y)\,dy\right|^2 = L(\tfrac{1}{2}, \pi\times\sigma)$ (Special function. integral)

$\mathcal{I}$ product of local integrals

$\Phi \in \pi$, $\phi\in\sigma$.

Idea: Can move ϕ, Φ in the rep π, σ.

~~Idea: Step up.~~

Idea: ~~[illegible]~~ Add up by on the basis of σ.

$$\int_{[H]} |\Phi(y)|^2\, dy = \sum_{\phi} L(\tfrac{1}{2}, \Phi\times\phi)\times \mathcal{I}_\phi$$

Choose Φ microlocalised (at $\lambda\in\mathfrak{g}^*$)

Show: $|\Phi|^2$ equidistributes on $[H]$.

So LHS $\sim$ vol$([H])$.

$\mathcal{I}$ also analysed using this choice

~~Rem~~ Φ.

Orbit method: Construct Φ using bump functions.

$\pi \longleftrightarrow$ Orbit $\mathcal{O}(\pi)$

Basis of Microlocalised vectors $v_i \longleftrightarrow$ Partition $\mathcal{P}$ of $\mathcal{O}(\pi)$ into vol(1) patches.

Op(a) projecting onto Φ $\longleftarrow\!\shortmid$ Bump function a

Orbit method

G = Lie grp. , g Lie alg, $\check{g} = ig^* = \mathrm{Hom}(g, i\mathbb{R})$

Co-adjoint orbit: G orbit $\mathcal{O}$ on g^*

induced by $\mathrm{Ad}_g : g \to g$

$\downarrow$

$(\mathrm{Ad}_g)^* : g^* \to g^*$

Kirillov: $\mathrm{Irr}(G) \xleftrightarrow{\text{approx bij}}$ coadjoint orbits of g^*

$\pi \longleftrightarrow \mathcal{O}_\pi$, symplectic form

$\mathrm{tr}\, \pi(f) \longleftrightarrow \int_{\mathcal{O}_\pi} f(\dots)\, d\xi$ $\uparrow$ symplectic measure.

f test func. $\in C_c^\infty(G)$

Ex $so(3)$: $g, g^* \cong \mathbb{R}^3$ Action of G = rotation on $\mathbb{R}^3$

Orbits : Spheres of rad r, $r \geq 0$

Irrep V_d of $su(2)$ on $so(3)$ $\longleftrightarrow$ $\mathcal{O}_r$ as above.

$d \in \mathbb{N}$ $\quad r \in \mathbb{R}^{\geq 0}$

approx inv.

Ex $G = PGL_2(\mathbb{R})$ $\quad g^*$ = Traceless 2×2 matrices.

Orbits
- Origin. (degenerate)
- One sheeted hyperboloid $\mathcal{O}^+(r) = \{x^2+y^2-z^2 = r^2\}$
- Two sheeted hyperboloid  $\mathcal{O}^-(r)$: $\{x^2+y^2-z^2 =$

$\mathcal{O}^+(\mathfrak{g}) \leftrightarrow$ Principal series

$\mathcal{O}^-(k) \leftrightarrow$ Discrete series

$f \in C_c^\infty(G)$. $\quad \pi(f) = \int_{g \in G} f(g)\,\pi(g)\,dg$.

f described using $\quad a \in S(\mathfrak{g}^*)$

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\exp"] \arrow[dr, "\bar f"'] & G \arrow[d, "f"] \\ & \mathbb{C} \end{tikzcd}$$

$f : G \to \mathbb{C} \quad \rightsquigarrow \quad \bar f : \mathfrak{g} \to \mathbb{C} \xrightarrow[f = Op(a)]{\text{Take FT}} a \in S(\mathfrak{g}^*)$

Kirillov formula: $\quad \mathrm{tr}(Op(a)) \approx \int_{\mathcal{O}_\pi} a\, d\omega.$

$\mathcal{O}_\pi = \coprod P_\tau \qquad P_\tau$ vol 1 $\qquad$ concentrated at $\tau \in \mathfrak{g}^*$

$\rightsquigarrow \quad V_\pi = \bigoplus \mathbb{C} v_\tau$

st. $\underbrace{Op(a)\, v_\tau}_{\pi(f) v_\tau} \approx a(\tau) v_\tau$

Microlocalized.
(simultaneously diagonalised)

Step 1) Refined GGP: $\sum_{v \in B(\pi)} \sum_{u \in B(\sigma)} \left(\int_{[H]} \theta_p(a) v \cdot \bar{u} \right) \left(\int_{[H]} \bar{v} u \right)$

$\theta_p(a) = \pi(f)$ test q.

$= L(\pi \times \sigma, \frac{1}{2}) \underbrace{H_{\pi, \sigma}(a)}$

Product of local integ.

Step 2) $\sum_{\sigma \in \mathcal{F}_T}$ or $\sum_{\pi \in \mathcal{F}_T}$ families $\mathcal{F}_T$ of π, or σ, short family.

$C(\pi) \in [T, T+1]$

Done by choosing a st $H_{\pi, \sigma}(a)$ detects $\mathcal{F}_T$

get: $\frac{1}{|\mathcal{F}_T|} \sum_{\sigma \in \mathcal{F}_T} L(\pi \times \sigma, \frac{1}{2}) \sim$ Expected const.

$\text{or } \pi \in \mathcal{F}_T$