Talk - Introduction to Subconvexity

~~Hardy-Littlewood~~ <u>History</u>

(0) Convexity bound : $|\zeta(\tfrac{1}{2}+it)| \ll (1+|t|)^{1/4}$

Phragmen - Lindelof, convexity arguments

(i) Hardy-Littlewood-Weyl (1921) : $|\zeta(\tfrac{1}{2}+it)| \ll_{\varepsilon} (1+|t|)^{1/6+\varepsilon}$

GL_1 (Weyl differencing)

~~Dirichlet characters~~

General problem: $L(\pi,s) \ll \underbrace{Q(\pi,s)}_{\text{conductor (analytic + alg. complexity)}}^{1/4-\delta}$ $\quad \delta>0$

q-aspect: Arith cond.
s-aspect: Analytic cond.

(ii) Dirichlet L-func (Burgess) : $L(\chi,s)$ χ primitive char, cond q

GL_1. $L(\chi,s) \ll_{\varepsilon,s} q^{1/4(1-\frac{1}{4})+\varepsilon}$ $\quad = 3/16$ $\quad$ cube free

(q-aspect) (s fixed)

(Method: Unique, not generalizable)

~~(Conrey-Iwaniec $L(\chi,s) \ll_{\varepsilon,s} q^{1/6+\varepsilon}$ χ quad char mod q~~)

(iii) Maass forms (DFI) : f primitive Hecke Maass cuspform - eigenvalue $\lambda = \tfrac{1}{4}+t_f^2$

1993 $\quad$ on $\Gamma_0(D)$

GL_2

$L(s,f) \ll (|t_f|+|s|)^{10}\, D^{1/4-\frac{1}{23041}}$ $\quad \mathrm{Re}(s)=\tfrac{1}{2}$

q aspect

Method : - Moment method + Amplification.
- δ-symbol method (Shifted conv. problem)

(iv) Conrey-Iwaniec (2000) : $L(\chi,s) \ll_{\varepsilon,s} q^{1/6+\varepsilon}$ $\quad$ to improve Weyl bd

χ <u>quad char</u> mod q

GL_1

Method: Moment method + Spectral theory $\quad$ q aspect

(ix) Munshi (2015): GL_3 automorphic form π, level 1

($SL_3(\mathbb{Z})$ Maass form)

$$L(\tfrac{1}{2}+it, \pi) \ll_\varepsilon (1+|t|)^{3/4 - 1/16 + \varepsilon}$$

Method: δ-symbol method

Holowinski - Nelson (2018): Simplified above method.
& Lin (2021) χ Dirichlet char, cond q

$$L(\phi \times \chi, s) \ll_{\phi} Q(\chi, s)^{3/4 - 1/36 + \varepsilon}$$

(x) Munshi (2021): $GL_3 \times GL_2$ Rankin - Selberg s-aspect.

ϕ Maass form, g Mod form.

$$L(\phi \times g) \ll_{\varepsilon, \phi, g} |s|^{3/2 - 1/51 + \varepsilon}$$

(s-aspect)

Method: δ-symbol method

(xi) Blomer Buttcane (2020): SCP for GL_3 reps

Method: Moment Method + Amplification.

(Uses KTF)

4^{th} moment.

Formalisms

- L-func $L(s,f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s}$

$$= \prod_p \left((1-\alpha_{f,1}(p)p^{-s}) \cdots (1-\alpha_{f,k}(p)p^{-s}) \right)^{-1}$$

k = degree of L-func.

- Completed: $\Lambda(s,f) = q_f^{s/2} \left(\prod_{i=1}^{k} \pi^{-s/2} \Gamma\left(\frac{s+\mu_i}{2}\right)\right) L(s,f)$

$(\mu_1, \dots, \mu_k) \in \mathbb{C}^k$, Langlands parameter.

q_f : Alg Conductor $\in \mathbb{N}$

(Give example : $\zeta(s)$, $L(\chi, s)$, Maass Form)

$$\Lambda(s,f) = \varepsilon(f) \overline{\Lambda(1-\bar{s},f)} \qquad |\varepsilon(f)|=1$$

- ~~Analytic~~ Conductor : $q(s,f)$ or $C(s,f)$ $= q_f \underbrace{\prod_{i=1}^{k} (|s+\mu_j|+1)}$

Measures the complexity or q_∞

- Approx FE: $\sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} V_s\left(\frac{n}{\sqrt{q}}\right) + \varepsilon(s,f) \sum_{n=1}^{\infty} \frac{\overline{\lambda_f(n)}}{n^{1-s}} V_{1-s}\left(\frac{n}{\sqrt{q}}\right)$

$\approx L(s,f)$

2 sums of length $\sqrt{q(s,f)}$

- Convexity bd: Trivial estimate: $\lambda_f(n) \ll_\varepsilon n^\varepsilon$

Plug into approx FE:

$$L(\tfrac{1}{2}+it,f) \ll \left(q\left(\tfrac{1}{2}+it,f\right)\right)^{1/4+\varepsilon}$$

$\left(\text{~~Because~~ } \sum_{n=1}^{\sqrt{q}} \frac{1}{\sqrt{n}} = q^{1/4}\right)$

(Also can be proved using Phragmen Lindelof (see Goldfeld (L)) Thm 8.2.3)

- Subconvexity:

$$\boxed{\text{Rankin selberg} \quad L(\pi_1 \times \pi_2, 1/2) \ll_{\pi_2} C(\pi_1)^{\deg \pi_2/4 - \delta}}$$

3

Since: $Q(\pi_1, s)^{\deg \pi_2} \asymp_{\pi_2 \text{ fixed}} Q(\pi_1 \times \pi_2, s)$

Methods of Moments

Goal: $L(f \times \chi, 1/2)$ $\quad f$ modular form : FIXED

$\chi \pmod q$ dirichlet char.

Level aspect: $q \to \infty$

To show: $|L(f \times \chi, 1/2)| \ll_f q^{1/2 + o(1)}$

Step 1) $\sum_{\chi \bmod q} |L(f_\chi, 1/2)|^2 \ll q^{1+o(1)}$

averaging techniques.

(Use approx FE & unfold)

Thus, $|L(f \times \chi_0, 1/2)| \ll q^{1/2 + o(1)}$

Convex bd. Doesn't beat convexity

(Note: conductor $(f \times \chi) = q(f)\, q^2$ if coprime

So convexity bd is $O(q^{1/2+\varepsilon})$)

Step 2) Amplification (due to Iwaniec)

put ~~more weight on~~

$L(f \times \chi_0, s)$, other wts = +ve.

use orthogonality of characters, to detect χ_0

Amplifier $a(\chi) = \boxed{\left|\sum_{n \sim N} \chi(n)\, \overline{\chi_0(n)}\right|^2} = \begin{cases} \asymp N & \text{if } \chi = \chi_0 \\ \text{small} & \text{if } \chi \neq \chi_0 \end{cases}$

(related to $L(\chi\bar{\chi}_0, s)$

no pole as $\chi\bar{\chi}_0 \neq 1$)

$$\sum_{\chi \bmod q} |L(f \times \chi, 1/2)|^2\, a(\chi)$$

$N^2 |L(f \times \chi_0, 1/2)|^2 \leq$

In summary, $\mathcal{F}_Q$ = Family of L-func. / aut reps on GL_d
with analytic cond. in $[Q, 2Q]$

$Q \to \infty$.

$$|L(\pi_0, 1/2)|^{2K} \leq \sum_{\pi \in \mathcal{F}} |L(\pi, 1/2)|^{2K}$$

$$\approx |\mathcal{F}| \quad \text{using Moment estimates.}$$

(By GLH, expect each term $\ll 1$)

$$\Rightarrow |L(\pi_0, 1/2)| \leq |\mathcal{F}|^{1/2K}$$

Sub-Convexity = $Q^{1/4 - \delta}$.

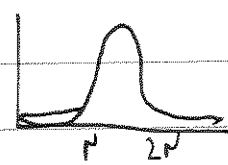
Thus we need, $|\mathcal{F}| < Q^{2K(\frac{1}{4} - \delta)}$

Options:
- Make k bigger (Higher moment harder to compute)
- Make $\mathcal{F}$ smaller (short interval estimates)
- Amplification. (Useful in the edge case)

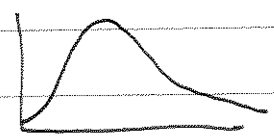
Hecke orbit approach (Venkatesh)

Approx FE: $\sum \frac{\lambda(n)}{n^s} V\left(\frac{n}{\sqrt{q}}\right) + \dots$

Fourier exp. (Whittaker) of Modular fn. $\phi(g) = \sum \lambda(n) W_\phi(y) e^{2\pi i n x}$

V: 

Cutoff.

W: 

Exp decay

Kirillov model: Parametrize aut fcns in $\pi \in GL_2$ by $C_c^\infty(\mathbb{R}^\times)$

Can replace W with V
ϕ " $\phi' \in \pi$

Thus express values of L-function as values of $\phi' \in \mathfrak{h}^2$
(Use fact that: On $\Gamma\backslash\mathfrak{h}^2$, ϕ' values eq-idistribute.)

(Other eg: $Z(s,\phi) = \int_{\mathbb{A}^\times} W_\phi\begin{pmatrix} y & \\ & 1\end{pmatrix} |y|^{s-1/2} d^\times y$ Zeta integrals Godement-Jacquet L-func.)

Subconvexity by Automorphic periods

Bernstein & Reznikov (2010), Michel-Venkatesh (2010), Paul Nelson (2021-) & Nelson-Venkatesh

[Goal: Bound $L(\frac{1}{2}, \pi_1 \otimes \pi_2 \otimes \pi_3) \ll_{\pi_1,\pi_2} C(\pi_3)^{1/4-\delta}$]

1. Bernstein - Reznikov: Used Ichino's formula / Watson's formula.

Unit vectors $v_j \in \pi_j$ AR

$$\frac{L(\frac{1}{2}, \pi_1 \otimes \pi_2 \otimes \pi_3)}{L(1, Ad^\circ \pi_1) L(1, Ad^\circ \pi_2) L(1, Ad^\circ \pi_3)} \underbrace{\mathcal{L}_\infty(v_1,v_2,v_3)}_{\text{Matrix coeff integral}} = \underbrace{\left| \int_{g \in \Gamma \backslash SL_2(\mathbb{R})} v_1 v_2 v_3(g) dg \right|^2}_{\text{Period}}$$

$\Gamma = PGL_2(\mathbb{Z})$

⚠ For v_1, v_2, v_3 Maass forms: $\mathcal{L}_\infty$ = Gamma factors, exp decay.

$$\Rightarrow L(\tfrac{1}{2}, \pi_1 \otimes \pi_2 \otimes \pi_3) \ll_{\pi_1,\pi_2} \text{Period} \times \exp(\tfrac{\pi}{2}|\lambda_3|)$$

λ_3 eigenvalue spectral par. of π_3

Thus very difficult to bd period.
Also period very difficult to evaluate.

Idea: > Choose nice test vectors v_1, v_2, v_3 to make $\mathcal{L}_\infty \gg 1/Q$

$Q = C(\pi_3)$

> Evaluate period using rep theory.

$\mathbb{Q}$

B-R: $\displaystyle\sum_{T\le t_3\le T+T^{1/3}} L(\tfrac{1}{2}, \pi_1\times\pi_2\times\pi_3) \ll T^{5/3+\varepsilon}$

(Hard analysis, hypergeom func.)
Need softer methods.

2. <u>Michel - Venkatesh</u> (GL_2) ~ Softer methods.

Pairs $H \subseteq G$ $\pi \in \mathcal{A}(G)$, $\sigma \in \mathcal{A}(H)$ aut reps.

subgrp.   reductive grp $\phi \in \pi$, $\psi \in \sigma$.

$$\mathcal{P}(\phi,\psi) := \int_{[H]} \phi(h)\psi(h)\,dh \qquad [H] := H(\mathbb{A})/H(\mathbb{Q})$$

<u>automorphic period</u>

For certain pairs (G,H), GGP pairs especially

$$\frac{\left|\int_{[H]} \phi(h)\psi(h)\,dh\right|^2}{\langle\phi,\phi\rangle\langle\psi,\psi\rangle} = \underbrace{C(\pi,\sigma)}_{\substack{\text{arithmetic}\\ \text{info}}}\, \Lambda(\pi,\sigma,\tfrac{1}{2}) \underbrace{\prod_v \int_{H(\mathbb{Q}_v)} \frac{\langle h\phi_v,\phi_v\rangle\langle h\psi_v,\psi_v\rangle}{\langle\phi_v,\phi_v\rangle\langle\psi_v,\psi_v\rangle}\,dh}_{\substack{\text{Matrix coeff integral}\\ \text{to be bdd locally}}}$$

$> Q(\pi,\sigma,\tfrac{1}{2})^{o(1)}$

Example: (Jacquet-Langlands, 1970) χ idelic character $\mathbb{A}^\times/\mathbb{Q}^\times \to \mathbb{C}$

(GL_2, A) $\pi \in \mathcal{A}_0(GL_2)$ CAR

$\phi = \otimes \phi_v \in \pi$ be pure tensor aut form.
(Whittaker model?)

Then, $$\mathcal{P}(\phi,\chi) := \int_{[A]} \phi(h)\chi(h)\,dh = \Lambda(\pi\times\chi,\tfrac{1}{2}) \prod_v \frac{\int_{A(\mathbb{Q}_v)} \phi_v(h)\chi_v(h)\,dh}{L_v(\pi\times\chi,\tfrac{1}{2})}$$

$$A = \left\{ h(t) = \begin{pmatrix} t & 0 \\ 0 & 1\end{pmatrix} \right\} \le GL_2$$

$\chi(h) := \chi(\det h) = \chi(t$

Step 3) Picking out a single $\varphi_0 \in \sigma$

(GL_2, A) case: χ_0 primitive mod q, π everywhere unram ($\phi_0 \in \pi$)

$$|P(\phi, \chi_0)|^2 \leq \sum_{\chi \in \widehat{[A]}} |P(\phi, \chi)|^2 = \int_{[A]} |\phi(h)|^2 \, dh$$

$$\frac{|L(\pi \times \chi_0, 1/2)|^2}{q} \qquad \ll 1$$

Choice of $\phi \in \pi$ or $\varphi_0 \in \sigma$

Evaluated trivially

(Picking ... trace formula)

Usually get convexity bound, via generous choice of test vectors. Amplify for subconvexity.

<u>Use of relative trace formula</u> to compute moments. $\sum_{\phi \in B(\pi)} P(\phi, \varphi_0)$

(Horizontal direction)

$f \in C_c^\infty(G(\mathbb{A}))$

$$K_f(x,y) = \sum_{\gamma \in G(\mathbb{Q})} f(x^{-1}\gamma y) \quad \text{automorphic kernel}$$

for convolution map $\phi \mapsto R(f)\phi = \int_{G(\mathbb{A})} f(g)\phi(xg)\,dg$ on $L^2([G])$

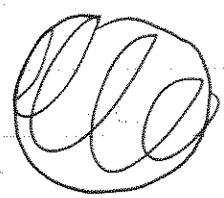
$$K_f(x,y) = \sum_{\pi \in A(G)} \sum_{\phi \in B(\pi)} R_f(\bar{\phi})(x)\, \phi(y)$$

~~Matrix coeff:~~ $Q(\phi) = \int_{[H]} \langle \pi(h)\phi, \phi \rangle$

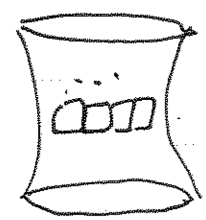
~~Trace evaluated~~ to $\mathcal{O}_\pi$

Eg: $G = GL_2$, $H = SO(2)$

Microlocalisation:



Principal series π_λ $GL(2)$ $\longleftrightarrow$ 1 sheeted Hyperboloid



H acts by rotation.

$\mathcal{O}_\pi = \bigcup P_i$

vector v_j Ortho basis $\mathcal{B}(\pi)$ $\longleftrightarrow$ Partitions P_i (ball of vol 1) microlocalised at $\tau_i \in P_i$

$\pi(g) v_i \approx \chi_{\tau_i}(g) V_i$

g small

χ_{τ_i} character on G.

Projection: weight $\frac{1}{\mathrm{vol}(\cdot)}$

Matrix coeff.

$\int_K \langle \pi(k) v_j, v_j \rangle$

$\approx \int_{(0,\varepsilon)} \underbrace{e^{i c_j \phi}}_{(a_j, b_j, c_j) \in \mathcal{O}_\pi} \underbrace{d\phi}_{\text{Sympl meas}}$

Talk 2 - Orbit ~~method~~ & ~~Subconvexity~~ seminar.

~~Orbit method & SGP~~ Micrololocal analysis

Maass forms ϕ_i, $\int_{\Gamma\backslash H} \phi_1\phi_2\phi_3$ = Hard to analyse. (Good : difficult hypergeom fn)

Basic problem (Toy model) : $L^2(S^2) = \bigoplus_{\ell \in \mathbb{N}} V_\ell$ $\dim V_\ell = 2\ell+1$

(Spherical harmonics irrep of SO(3))

V_ℓ = Harmonic ($\Delta f = 0$) poly hom. in x,y,z deg ℓ.

$V_\ell = \bigoplus_m Y_{\ell,m}$ st $Y_{\ell,\ell} = (x+iy)^\ell$

(double cover of $so(2)$)

$$\int_{S^2} Y_{\ell_1,m_1} Y_{\ell_2,m_2} Y_{\ell_3 m_3} = \boxed{?}$$

Very hard problem. $G = SO(3)$, $\mathfrak{g} = \mathbb{R}^3$, $\hat{\mathfrak{g}} = \mathbb{R}^3$

Step 1) Idea: Chop up $f = \sum_{\alpha \in \mathbb{R}^n \oplus \hat{\mathbb{R}}^n} f_\alpha$ (position sp, freq sp) f_α localised in space & frequency. $f_\alpha, \hat{f}_\alpha$ reasonable behavior (can't be completely loc due to Heisenberg uncertainty)

Step 2) Decompose $Y_{\ell_i,m_i} = \int c_\alpha v_\alpha$ Microlocalised at α

~~Microlocal~~:

$$\int_{S^2} Y_{\ell_1,m_1} \cdots Y_{\ell_3,m_3} = \int_{\alpha,\beta,\gamma} \cdots \int_{S^2} v_\alpha v_\beta v_\gamma$$

~~Approx~~ Much easier to evaluate.

$= 0$ unless $\ell_1\alpha + \ell_2\beta + \ell_3\gamma = 0$.

Orbit method

G = Lie grp., g Lie alg, $\check{g} = g^* = \mathrm{Hom}(g, \mathbb{R})$

Co-adjoint orbit: G orbit $\mathcal{O}$ on g^*

induced by $Ad_g : g \to g$

$\downarrow$

$(Ad_g)^* : g^* \to g^*$

Kirillov: $\mathrm{Irr}(G) \xleftrightarrow{\text{approx bij}}$ coadjoint orbits of g^*

$\pi \longleftrightarrow \mathcal{O}_\pi$, symplectic param

$\mathrm{tr}\, \pi(f) \longleftrightarrow \int_{\mathcal{O}_\pi} f(\dots)\, d\xi$ ↑ symplectic measure.

f test func. $\in C_c^\infty(G)$

Ex $SO(3)$: $g, g^* \cong \mathbb{R}^3$ Action of G = rotation on $\mathbb{R}^3$

Orbits : Spheres of rad r, $r \geq 0$

Irrep V_d of $su(2)$ on $so(3)$ $\longleftrightarrow$ $\mathcal{O}_r$ as above.

$d \in \mathbb{N}$, $r \in \mathbb{R}^{\geq 0}$

approx inv.

Ex $G = PGL_2(\mathbb{R})$ g^* = Traceless 2×2 matrices.

Orbits
- Origin. (degenerate)
- One sheeted hyperboloid $\mathcal{O}^+(r) = \{x^2+y^2-z^2=r^2\}$

- Two sheeted hyperboloid $\mathcal{O}^-(k) = \{x^2+y^2-z^2 = -k$

$\mathrm{Op}(a) = \pi(f)$ test q.

Step 1) Refined GGP: $\displaystyle\sum_{v \in B(\pi)} \sum_{u \in B(\sigma)} \left(\int_{[H]} \mathrm{Op}(a)\, v \cdot \bar{u} \right) \left(\int_{[H]} \bar{v}\, u \right)$

$$= L(\pi \times \sigma, \tfrac{1}{2})\, \underbrace{H_{\pi,\sigma}(a)}_{}$$

product of local integrals.

Step 2) $\displaystyle\sum_{\sigma \in F_T}$ or $\displaystyle\sum_{\pi \in F_T}$ families F_T of π, or σ, short family. $c(\pi) \in [T, T+1]$

Done by choosing a st $H_{\pi,\sigma}(a)$ detects F_T

get: $\displaystyle\frac{1}{|F_T|} \sum_{\substack{\sigma \in F_T \\ \text{or } \pi \in F_T}} L(\pi \times \sigma, \tfrac{1}{2}) \sim$ Expected const.

Alan Zhao - MV paper. Test functions

~~Test functions in Michel-Venkatesh~~

Notation.

F local field, $K_v/\mathbb{Q}_p$ finite ext.
$\mathbb{R}$ or $\mathbb{C}$

$GL_2 \supseteq B = \{\text{Upper } \Delta\} \supseteq A = \{ \begin{pmatrix} y & \\ & 1 \end{pmatrix} \mid y \in F^\times \}$
$\cup\!\text{l}$ $\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \overset{\shortparallel}{a(y)}$

$Z = \{ \begin{pmatrix} z & \\ & z \end{pmatrix} \mid z \in F^\times \}$

$U = \{ u(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in F \} \subseteq B$

~~Models of aut rep $\pi \subseteq A(GL_2(F))$ (assume π irreducible)~~

π be local component at place v of some global aut. rep
($V \cong \otimes V_v$ tensor product thm)

Let π has $\begin{cases} \text{Whittaker model} & W(\pi, \psi) \subseteq \{ f(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g) = \psi(x) f(g) \} \\ \text{Kirillov model} & K(\pi, \psi) \subseteq C_c^\infty(F^\times) \end{cases}$

$V_f \in \pi \rightsquigarrow f$ in Kirillov model
$\rightsquigarrow W_f$ in Whittaker model

st $\langle V_f, V_f \rangle_\pi = \int |f(y)|^2 \, dy$

Triple product functionals : π_1, π_2, π_3 local parts of aut reps (at v)

& assume $\chi_1\chi_2\chi_3 = 1$ (when χ_i central char of π_i)

$$L : \pi_1 \otimes \pi_2 \otimes \pi_3 \to \mathbb{C}$$ G-invariant map

$$\Big\downarrow$$

$$\tilde{I}_{\pi_2} : \pi_1 \otimes \pi_3 \to \tilde{\pi}_2$$ (contragredient)

$$\Big\downarrow$$

$$I_{\pi_2} : \pi_1 \otimes \pi_3 \to \pi_2$$ (as $\tilde{\pi}_2 \cong \pi_2$ in GL_2)

(check)

Take $L = \begin{cases} L_W(x_1, x_2, x_3) = \displaystyle\int_{G(F)/Z(F)} \langle g x_1, x_1\rangle_{\pi_1} \langle g x_2, x_2\rangle_{\pi_2} \langle g x_3, x_3\rangle_{\pi_3} \, dg \\ \qquad x_i \in \pi_i \\ \qquad \text{This is why you need } \chi_1\chi_2\chi_3 = 1 \\ L_{RS}(x_1, x_2, x_3) = \displaystyle\int_{G(F)/N(F)Z(G)} W_{x_1} W_{x_2} W_{x_3}(g) \, dg \end{cases}$

Rankin-Sel

when W_{x_i} image of $x_i \in \pi_i$ in $W(\pi_i, \psi)$

Rem When everything is unramified & $v < \infty$ then $L_W = L_{RS}$

Non-arch case: χ_1, χ_2 will be newvectors in $W(\pi_1, \psi)$ & $W(\pi_1, \bar{\psi})$

χ_3 is an eigenfunction of action of $K_0(N) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathcal{O}_F) \mid$ "$N \mid c$" $\}$

Arch case: ϕ be bump function, compactly supported in set K_C

Choose $K_C \subseteq GL_2(\mathcal{O}_F) \cap B$ & $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_C \iff |c| \leq C^{-1}$

(want C to be large, a bit bigger than $c(\pi_1)$)

Kirillov Model

> $\chi_1 \leftrightarrow \phi$
>
> $\chi_2 \leftrightarrow \bar{\phi}$

clarify later:

> χ_3 defined by ϕ

$$\Phi(x,y) = \phi(C|x|)\, \phi(|y|-1)\, \chi_3(y)$$

Kir. Model

$$\chi_3 \longleftrightarrow \int_{F^\times} \Phi_3((0,t)g)\, \chi_3^{-1}(t)\, |t| \, d^\times t$$

Applications (Sec 4, 5 .. sketch) (~~...~~ To be covered)

~~Sketch~~

(Tool used: Plancherel formula. : Define next session)

Tool used: Amplification $\rightsquigarrow$ Geometrised version,
~~...~~ a measure is defined
that deals with it.

1) Pick global Π_3 (aut rep), Π_3 ~~...~~ be Eis series $Eis(\mathcal{I}(1,\chi_3))$

$\phi_1 = \otimes \phi_{1,v}$, $\phi_2 = \otimes \phi_{2,v}$, consists of the local test vectors chosen earlier.

$E_3 = \otimes E_{3,v} \in \Pi_3$
chosen earlier

Measure σ for amplification on $GL_2(\mathbb{A}_f)$

$\mathrm{supp}\,\sigma \subseteq GL_2(\mathbb{A}_f)$

Properties > $\phi_1 * \sigma = \lambda_1 \phi_1$ $\qquad \phi_1 \in \Pi_1$ (global)
some convolution.

- Use Cauchy Schwartz, this pops up in Subconvexity $\left(|\sigma|(f) = \int |f| \, d\sigma \right.$

$$|\lambda_1|^2 \boxed{|\langle \phi_1, \overline{\phi_2 E} \rangle|^2} \leq \int \langle \phi_2^u \bar{\phi}_2 , \bar{E}^u E \rangle_{reg} \, d|\sigma|^{(2)}$$

where $u \in GL_2(\mathbb{A}_f)$ fixed choice
& $\phi^u(g) = \phi(gu)$

Austin talk: Subconvexity for twisted L-fns of GL(2) mod forms

f modular form, wt k for $SL(2,\mathbb{Z})$, Hecke cusp form.

$f(z) = \sum b_m m^{\frac{k-1}{2}} e(mz)$, χ_0 primitive Dirichlet char mod q

$L_f(s,\chi_0) = \sum b_m \chi_0(m) m^{-s}$

$$\left(\frac{q}{2\pi}\right)^{s} \Gamma(\cdots)\Gamma(\cdots) L_f(s,\chi_0) = \varepsilon_{\chi_0}^2 i^k \Gamma(\cdots)\Gamma(\cdots) L_f(1-s,\bar{\chi_0})$$

$$\varepsilon_{\chi_0} = \tau(\chi_0) q^{-1/2}$$

$$\tau(\chi_0) = \sum_{a \,(\mathrm{mod}\, q)} \chi_0(a) e\left(\frac{a}{q}\right)$$

Convexity bd: $L_f(\tfrac{1}{2}+it,\chi_0) \ll q^{1/2+o(1)}$ $(|t|\cdots)$

$$Q(\pi_f \times \chi_0) = q^2$$

Main thm: $\mathrm{Re}(s) = 1/2$, $L_f(s,\chi_0) \ll |s|^2 q^{5/11} \tau(q)^2 \log q$

Main ideas: (i) Approx FE via Voronoi summation.

$$\sum_{m\in[M,2M]} b_m \chi_0(m) m^{-s} \xrightarrow{\text{Voronoi}} \text{Dual sum } \sum b_n \chi_0(n) n^{-s}$$

Sum of size M. Length q^2/M

Approx FE, $Re(s) = 1/2$

$$L_f(s, \chi_0) = \sum_m b_m \chi_0(m) m^{-s}$$

$$H = \sum_m b_m \chi_0(m) m^{-s} h(m)$$

$h(m)$ smooth, cpt support $[M, 2M]$

$|h^{(j)}(m)| \ll M^{-j}$

Voronoi summation: $\sum_m b_m e\left(\frac{am}{c}\right) F(m) = \sum b_n e\left(-\frac{\bar{a} n}{c}\right) \check{F}(n)$

$\leftarrow$ uses the fact that b_i are Fourier coeff of Modular form

$(a, c) = 1$, F smooth, cptly supp.

$a\bar{a} = 1 \pmod{c}$

$$\check{F}(y) = 2\pi i^k c^{-1} \int_0^\infty F(x) J_{k-1}\left(\frac{4\pi\sqrt{xy}}{c}\right) dx$$

$J_v(x) \ll \frac{1}{\sqrt{1+x}}$. For $x \gg 1$, decays rapidly

$$\underbrace{\sum_m b_m \chi_0(m) F(m)}_{\text{Length} \sim M} = \varepsilon_\chi \underbrace{\sum_n b_n \bar{\chi}_0(n) \check{F}(n)}_{\text{Length } q^2/M}$$

$$J_{k-1}\left(\frac{4\pi}{q} \sqrt{Mn}\right)$$

For $n \gg q^2/M$, it decays rapidly

$w(t): \mathbb{R} \to \mathbb{R} \quad w(0) = 0, \ \text{cptly supp.}, \quad \sum_{k=1}^{\infty} w(k) = 1$

$$\delta_k(n) := w(k) - w\left(\tfrac{n}{k}\right)$$

$$\delta(n) = \sum_{k \mid n} \delta_k(n) = \sum_k k^{-1} \sum_{h \,(\mathrm{mod}\, k)} e\left(\frac{hn}{k}\right) \quad \delta_k(n)$$

$$\Delta_c(n) := \sum_q q^{-1} \delta_{cq}(n)$$

$$\delta(n) = \sum_c c^{-1} \sum^*_{a \,(\mathrm{mod}\, c)} e\left(\frac{an}{c}\right) \Delta_c(n)$$

If $w(t)$ supp. on $\left[\frac{K}{2}, K\right]$, $n < \frac{N}{2}$, $K = \sqrt{N}$

Then, $\Delta_c(n) \ll K^{-1}$

$\Delta_c(n)$ is 0 for $c > K$

Thus $\delta(n)$ finite sum on c, $\sum_{c \le K} \sum^*_{a \,(\mathrm{mod}\, c)} e\left(\frac{an}{c}\right) \Delta_c(n)$

Input into $(*)$, $$S_h = \sum_c c^{-1} \underbrace{\sum^*_{a \,(\mathrm{mod}\, c)} \sum_{n_1, n_2} a_{n_1} \bar{a}_{n_2} \, e\left(\frac{a}{c}(n_1 - n_2 - h)\right) \Delta_c(n_1 - n_2 - h)}_{S_{h,c}}$$

(remember $a_n = \lambda_i * \chi(i)$)

Coming back,

$$S \le \phi(q) \sum_{h \equiv 0 (q)} S_h$$

$$S_h = \sum_c c^{-1} S_{h,c}$$

$$S_{h,c} = \Big(\sum_{l_1, l_2} \lambda_{l_1} \bar{\lambda}_{l_2}\Big) (LM)^{1/2} (h,c)^{1/2} \tau(c) c^{1/2}$$

$$= \Big(\sum_l |\lambda_l|\Big)^2$$ as ~~$l_1 \neq k$ gets cancelled~~.

Cauchy Schw.

Because of our choices, sum up $c \le K = \sqrt{N} = (2LM)^{1/2}$

$$S_h \ll \tau(h) (LM)^{3/4} (\log LM) \Big(\sum |\lambda_l|\Big)^2$$

Off diag: Sum over $h > 0$, $q | h$, $h < N = 4LM$

$$\sum_{\substack{h>0 \\ q|h}} S_h \ll \tau(q) (LM)^{7/4} (\log LM)^2 \Big(\sum |\lambda_l|\Big)^2$$

$$S_0 = \phi(q) \sum_a |a_n|^2 \ll M (\log M) \Big(\sum_l |\lambda_l|^2 \tau(l)\Big)$$

This gives upper bd for S.

~~Supremum Problem?~~ Orbit Method & Microlocal analysis

Ingredients

$$G = \text{Algebraic group } (GL_n, U_n, SO_n) \qquad [G] := G(\mathbb{Q})\backslash G(\mathbb{A})$$

RTF: $f \in C^\infty(G(\mathbb{A}))$

$$K_f(x,y) = \sum_{\gamma \in G(\mathbb{A})} f(x^{-1}\gamma y)$$

$$R(f) : L^2([G]) \to L^2([G])$$

$$\phi \mapsto \int_{g \in G(\mathbb{A})} f(g)\,\phi(xg)\,dg$$

$$= \int_{[G]} \underbrace{K_f(x,y)}_{\text{Kernel}} \phi(y)\,dy$$

$$K_f(x,y) = \underbrace{\sum_{\pi \in \mathcal{A}(G)}}_{\substack{\text{Actually} \\ \text{an integral}}} \underbrace{\sum_{\phi \in \mathcal{B}(\pi)}}_{\text{ONB}} R_f(\phi)(x)\,\overline{\phi(y)}$$

Now, for $\psi_0 \in \mathcal{A}([H])$ $\quad H \le G$. $\quad (H = GL_{n-1}, SO_{n-1}, U_{n-1})$

$$\iint_{[H]\times[H]} (\cdot)\,\overline{\psi_0(x)}\,\psi_0(y)\,dx\,dy \text{ on both sides}$$

Thus, $$\int_{[H]\times[H]} K_f(x,y)\,\overline{\psi_0}(x)\,\psi_0(y)\,dx\,dy = \sum_{\pi \in \mathcal{A}(G)} \sum_{\phi \in \mathcal{B}(\pi)} \overline{P(R(\bar{f})\phi, \psi_0)}\,P(\phi,\psi_0)$$

where $$P(\phi,\psi_0) = \int_{[H]} \phi(h)\,\psi_0(h)\,dh$$

$\Bigg[\Bigg[$ If $R(f)$ approx a proj. op. onto a family $\mathcal{F}$ of $A(G)$ $\Bigg]\Bigg]$

Thus. RTF:

$$\text{Spectral side} = \sum_{\pi \in A(G)} \sum_{\phi \in \mathcal{B}(\pi)} \overline{P(R(\bar{f})\phi, \varphi_0)}\, P(\phi, \varphi_0)$$

$$= \sum_{\pi \in \mathcal{F}} \sum_{\phi \in \mathcal{B}(\pi)} \underbrace{\overline{P(\phi, \varphi_0)}\, P(\phi, \varphi_0)}_{= |P(\phi, \varphi_0)|^2}$$

Apply GGP

$$= \sum_{\pi \in \mathcal{F}} \sum_{\phi \in \mathcal{B}(\pi)} \Lambda(\pi \otimes \sigma, 1/2) \prod_v Q_v(\phi_v, \varphi_v)$$

Avg over short families $\mathcal{F}$.

$$\text{Geom side} = \int_{[H]} \int_{[H]} \sum_{\gamma \in G(\mathbb{Q})} f(x^{-1}\gamma y)\, \varphi_0(x)\, \overline{\varphi_0(y)}\, dx\, dy$$

Main term $= \{\gamma \in H(\mathbb{Q})\}$ terms

Error term $= \{\gamma \in G(\mathbb{Q}) \setminus H(\mathbb{Q})\}$ terms.

Main term. $>$ Error term. get convexity bd.

Amplification: Choose f_p nicely, balance MT & ET $\Rightarrow$ Subconvex bd.

What is the orbit method

G = Lie group. $\mathfrak{g} = \mathrm{Lie}(G)$, $\mathfrak{g}^\wedge = \mathrm{Hom}(\mathfrak{g}, i\mathbb{R})$

$G \curvearrowright \mathfrak{g}$ by Ad (adjoint action), Dualise to $\mathfrak{g}^\wedge$ to get coadj action.

Coadj. orbit $\mathcal{O} \subseteq \mathfrak{g}^\wedge$ G-orbit for the coadjoint action.

Kirillov: $\{$Irred tempered G-reps$\}$ $\xleftrightarrow[\text{(approx)}]{\text{bij}}$ $\{\mathcal{O}$ coadjoint orbits$\}$

$$\pi \longleftrightarrow \mathcal{O}_\pi \quad \text{(symplectic structure) measure } \omega$$

Kirillov formula:

$\chi_\pi : G \to \mathbb{C}$ (Harish Chandra char.) $\longleftrightarrow$ Integration wrt ω.

$x \in \mathfrak{g}$ small: $$\chi_\pi(\exp(x)) = j(x)^{-1/2} \int_{\xi \in \mathcal{O}_\pi} e^{\langle x, \xi \rangle} \, d\omega(\xi)$$ (FT of symplectic measure)

(Jacobian of $\exp : \mathfrak{g} \to G$ st $j(0) = 1$)

($\langle x, \xi \rangle = \xi(x)$)

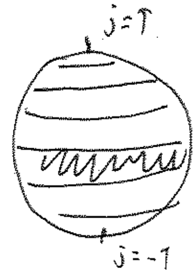
Harish-Ch char: For $f \in C_c^\infty(G)$, $\pi(f) = \int_{g \in G} \pi(g) f(g) \, dg$. trace given by $\chi_\pi \in C_c^\infty(G)$

Thm: Valid for G nilpotent (Kirillov)

compact (Kirillov, Weyl)

reductive, tempered (Rossmann)

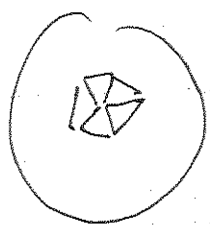
$$\dim \pi \longleftrightarrow \mathrm{vol}(\mathcal{O}_\pi)$$
$(g = 1)$



$\pi_T = \bigsqcup_{j=-T}^{T} P_j$

3 strips P_j for each j

Could have also taken balls instead of strips



tilings P_τ concentrated around a pt $\tau \in \mathcal{O}_\pi$

 P_τ

$G = SO(3)$, $\pi = \pi_T$ $\quad P_T \approx \mathcal{O}_\pi \cap \{\tau + O(T^{1/2})\}$

$$\updownarrow$$

$\{v_\tau\}$ ONB of microlocalised vectors

$\pi(e^x)\, v_\tau \approx \underset{\sim}{\langle x, \tau \rangle}\, v_\tau$ (will make it concrete soon)

KEY PROPERTY: v_τ microlocalised as above,
Matrix coeff of v_τ concentrate near centraliser $G_\tau = \{g \in G \mid g \cdot \tau \approx \tau\}$

Heuristic: $g v_\tau$ microlocalised at $g \cdot \tau$
Thus if g, $g \cdot \tau$ are far away $\langle g \cdot v_\tau, v_\tau \rangle = 0$
(as eigenvalues are different)

Sup-norm problem & the orbit method.

Supnorm problem (GL_2)

$\mathbb{H}$ = Upper half plane, $SL(2,\mathbb{R}) \curvearrowright \mathbb{H}$

$\Gamma_0(N) = \{ \gamma \in SL_2(\mathbb{Z}) \mid \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod N \}$

$f : \Gamma_0(N) \backslash \mathbb{H} \to \mathbb{C}$ bdd smooth function.

The L^∞-norm is $\|f\|_\infty = \sup_{z \in \mathbb{H}} |f(z)|$

The L^2-norm is $\|f\|_2 = \dfrac{1}{\mathrm{vol}(\Gamma_0(N) \backslash \mathbb{H})} \displaystyle\int_{\Gamma_0(N)\backslash\mathbb{H}} |f(x+iy)|^2 \frac{dx\,dy}{y^2}$

Thm: $N = p^{4n}$, $f : \Gamma_0(N) \backslash \mathbb{H} \to \mathbb{C}$ cuspidal, Hecke-Maass newform with trivial nebentypus.

$\Delta f = \lambda f$ eigenvalue λ.

For λ sufficiently large, $\|f\|_\infty \ll_{p,\varepsilon} (N\lambda)^{5/24+\varepsilon} \|f\|_2$

(Hybrid bound in N & λ level & spectral aspect)

Convexity bound : $\|f\|_\infty \ll_{N,\varepsilon} \lambda^{1/4+\varepsilon} \|f\|_2$
(Baseline bound) (Spectral aspect)

$\sup_{\lambda_f \in I} \|f\|_\infty \ll_{I,\varepsilon} N^{\beta+\varepsilon} \|f\|_2$
(spectral window) (Level aspect)

For N sq free, $\beta = 1/2$

$N = N_0^2$, $\beta = 1/4$

Setting up the problem

$G_\infty = GL_2(\mathbb{R})$, $\quad G_p = GL_2(\mathbb{Q}_p)$, $\quad G^{(\mathbb{A})} = GL_2(\mathbb{A})$

$K_\infty = SO(2)$ $\quad K_p = GL_2(\mathbb{Z}_p)$

$N = p^{4n}$, $\quad N_0 = p^{2n}$

$$K^*(N_0) = K_\infty \times \prod_{\ell \neq p} K_\ell \times K_{H,p}(2n, 2n)$$

where $K_{H,p}(2n, 2n)$

$$= \left\{ \begin{pmatrix} a & p^{2n} b \\ p^{2n} c & d \end{pmatrix} \;\middle|\; \begin{matrix} a,b,c,d \\ \in \mathbb{Z}_p \end{matrix} \right\} \cap GL_2(\mathbb{Q}_p)$$

$\Gamma_H(N_0, N_0) = SL_2(\mathbb{Z}) \cap \left\{ \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix} \bmod N_0 \right\}$

Strong approx: $\Gamma_H(N_0, N_0) \backslash \mathbb{H} \cong Z(\mathbb{A}) G(\mathbb{Q}) \backslash G(\mathbb{A}) / K^*(N_0) = X$

$[G] = Z(\mathbb{A}) G(\mathbb{Q}) \backslash G(\mathbb{A})$

$\left\{ \text{Hecke-Maass cuspidal newform } \phi : \mathbb{H} \to \mathbb{C} \text{ left } \Gamma_0(N)\text{-invariant} \right\}$

$\downarrow$ Translate.

$\left\{ \Phi : \mathbb{H} \to \mathbb{C} \text{ left } \Gamma_H(N_0, N_0) \text{ invariant} \right\}$

$\downarrow$ Adelise

$\left\{ \phi : G(\mathbb{A}) \to \mathbb{C} \text{ automorphic form.} \right\}$

$\downarrow$ generates

$\pi \in L^2([G])$ with unique $K^*(N_0)$-invariant vector ϕ

irred CAR

(locally ϕ_v is a newvector. $K_H(2n, 2n)$-invariant)

p-adically:

$K_H(m, m') = \begin{pmatrix} a & p^{m'} b \\ p^{m} c & d \end{pmatrix}$

conjugate by $\begin{pmatrix} p^{-m'} & \\ & 1 \end{pmatrix}$ to get

$K_0(n) = \begin{pmatrix} a^* & b^* \\ p^{n} c^* & d^* \end{pmatrix}$

$n = m + m'$

~~Records~~

$H = K^*(N_0)$ $\quad Z_H = Z \cap H$ compact. normalised measure, st $\mathrm{vol}(Z_H) = 1$.

π = CAR of $GL_2(\mathbb{A})$, trivial central character, level $N = p^{4n}$

Assume: π tempered & spherical at ∞.
with ~~spectral~~ parameter $T \in \mathbb{R}$ large.
(where $\Delta \phi = \underbrace{1/4 - s^2}_{\lambda} \phi$

tempered $\Rightarrow s = iT$)

~~Classical~~ Lift(ϕ) = Cuspidal Hecke-Maass newform that is $K_0(N)$ invariant
(Adelic) $\phi' = \pi(a(N_0))\phi$ is $K^*(N_0)$-invariant
where as we saw before
$(a(N_0)^{-1} K^*(N_0)_{fin}\, a(N_0) = K_0(N))$

$K^*(N_0)$-inv. vectors in π spanned by ϕ'
More convenient to work with

Period integral: $\ell(\Phi) = \frac{1}{\mathrm{vol}(H)} \int_H \Phi(h)\, dh$ $\qquad \Phi \in \pi$ (on L)

Claim: $|\ell(\Phi)|^2 = |\phi'(1)|^2\, Q(\Phi)$

where $Q(\Phi) := \frac{1}{\mathrm{vol}(H)} \int_H \langle \pi(h)\Phi, \Phi \rangle\, dh$

$Q(\Phi) = Q_\infty(\Phi_\infty) \prod_p Q_p(\Phi_p)$ for factorisable Φ.

$Q_v(\Phi_v) = \frac{1}{\mathrm{vol}(H_v)} \int_{H_v} \langle \pi(h_v)\Phi_v, \Phi_v \rangle\, dh_v$

Matrix coeff

Pre-trace inequality: $f \in C_c^\infty(G)$, $H \leq G$. (general H) $\pi \in L^2([G])$

$$\sum_{\Phi \in \mathcal{B}(\pi)} \underbrace{\left| \int_H \pi(f) \Phi \right|^2}_{\ell(\pi(f)\Phi)} \leq \iint_{x,y \in H} \sum_{\gamma \in G'(\mathbb{Q})} (f * f^*)(x^{-1}\gamma y)\, dx\, dy$$

Pf:

Apply to period formula; to f_g, H^g $f_g(x) := f(g^{-1} x g)$ $H^g =$

$f_g * f_g = (f * f^*)_g$

$$\sum_{\varphi \in \mathcal{B}(\pi)} |\ell_g(\pi(f_g) \cdot \Phi)|^2 = |\phi'(g)|^2 \sum_{\Phi \in \mathcal{B}(\pi)} \underbrace{Q_g(\pi(f_g)\pi(g)\Phi)}_{= Q(\pi(f)\Phi)}$$

Main Trace inequality:

$$|\phi'(g)|^2 \sum_{\varphi \in \mathcal{B}(\pi)} Q(\pi(f)\Phi) \leq \frac{1}{\mathrm{vol}(H)} \int_{x,y \in H} \sum_{\gamma \in G'(\mathbb{Q})} (f * f^*)(x^{-1} g^{-1} \gamma g y)\, dx\, dy$$

Thus,

$$\sum_{\phi \in B(\pi)} Q(\pi(f)\phi) = \sum_{v \in B(\pi_\infty)} Q_\infty(\pi(f_\infty)v)\; Q_p(v_p)$$

$$Q_p(v_p) = \zeta_p(1)\, p^{-n} \qquad \text{(calculation done later)}$$

$$I_\gamma(f_p * f_p) = \frac{1}{\mathrm{vol}(H_p(2n))^2} \int_{H_p(2n)} \int_{H_p(2n)} [f * f](x^{-1}\gamma y)\, dx\, dy$$

$$H_p^{(2n)} = K_p(2n, 2n)$$

$$\mathrm{supp}(f) \supseteq Z(\mathbb{A})(\mathcal{U} \times K_f)$$

$$\left(Z(\mathbb{A}) \setminus Z(\mathbb{A})(\mathcal{U} \times K_f) \right) \cap PGL_2(\mathbb{Q}) = SL_2(\mathbb{Z}) \cap \mathcal{U}$$

(strong approx)

Thus,

$$|\phi'(g)|^2 \cdot \frac{\zeta_p(1)}{p^n} \sum_{v \in B(\pi_\infty)} Q(\pi(f_\infty)v)$$

$$\leq \sum_{\gamma \in \Gamma \cap \mathcal{U}} I_{g^{-1}\gamma g}(f_p * f_p) \int_{x, y \in K_\infty'} (f * f^*)(x^{-1} g^{-1} \gamma g y)\, dx\, dy$$

$$\gamma = 1: \quad \int_{K_\infty'} (f_\infty * f_\infty^*)(y)\, dy \ll T^{1/2 + \varepsilon}$$

GIT quotients

Recap: $G = PGL_2(\mathbb{R})$

$g = sl_2(\mathbb{R}) = sp(A, B, C)$

$$A = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$g^* = \mathrm{Hom}_{\mathbb{R}}(g, \mathbb{R})$, $\quad g^*_{\mathbb{C}} = g^* \otimes_{\mathbb{R}} \mathbb{C}$

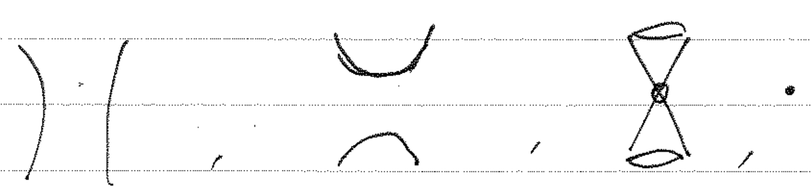
$g^\wedge = \mathrm{Hom}(g, \mathbb{C}^{(1)})$, $\quad ig^* = \mathrm{Hom}(g, i\mathbb{R})$

> $g^\wedge \cong ig^*$ by exponentiating

> $g \cong g^*$ by trace pairing. Thus
$X \mapsto \mathrm{tr}(X, \cdot)$

> Coadjoint action by G on $g^*_{\mathbb{C}}$

> Constructed coadjoint orbits on g^*



$\mathcal{O}_{TA}$ $T>0$ $\quad$ $\mathcal{O}_{TC}$, $T>0$ $\quad$ $\mathcal{O}_{B+C}$ $\quad$ $\mathcal{O}_0$

$\{a^2+b^2-c^2 = T\}$ $\quad$ $\{a^2+b^2-c^2 = -T\}$ $\quad$ (det 0)

Coordinates $(a, b, c) \in \mathbb{R}^3 \longleftrightarrow aA + bB + cC \in g^*$

$\updownarrow$ trace pairing

$g^*_{\mathbb{C}}$

GIT quotient (formal)

1) $G = GL_2(\mathbb{R})$ $\quad \mathfrak{g} = Mat_{2\times 2}(\mathbb{R}) = \left\{ \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \right\}$

$$\mathfrak{g}^*_{\mathbb{C}} = Mat_{2\times 2}(\mathbb{C})$$

$R_{\mathbb{C}}^G := \mathbb{C}[\mathfrak{g}^*_{\mathbb{C}}]^G$ poly in $\mathfrak{g}^*_{\mathbb{C}}$ invariant under G-action

Coadj. action $\equiv$ conjugation

$\Rightarrow R_{\mathbb{C}}^G = \mathbb{C}[x_1, x_2, x_3, x_4]^G$

$$\left.\begin{array}{l} \det(x) = x_1 x_4 - x_3 x_2 \\ tr(x) = x_1 + x_4 \end{array}\right\} \in R_{\mathbb{C}}^G$$

Can show: $R_{\mathbb{C}}^G = \mathbb{C}[\det, tr]$ ((Chevalley's restriction thm)

GIT quotient $= Hom(R_{\mathbb{C}}^G, \mathbb{C}) = [\mathfrak{g}^*_{\mathbb{C}}]$

Thus in our case, $[\mathfrak{g}^*_{\mathbb{C}}] \cong \mathbb{R}^2$

$$\begin{pmatrix} \det \mapsto p \\ tr \mapsto q \end{pmatrix} \longmapsto (p, q)$$

2. $G = PGL_2(\mathbb{R})$

$R_{\mathbb{C}}^G = \mathbb{C}[\det]$ since $tr = 0$.

Thus GIT quotients determined by value of det.

Case of $G = GL(n, \mathbb{R})$

$$\mathfrak{g} = \mathfrak{gl}_n(\mathbb{R}) \cong \mathfrak{g}^*_{\mathbb{C}}$$
$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{gl}_n(\mathbb{C}) \cong \mathfrak{g}^*_{\mathbb{C}}$$

(coadj action = conjugation.

$M \in \mathfrak{gl}_n(\mathbb{C})$.

Char poly = $\det(M - \lambda X)$

$= X^n + c_{n-1}(m_{11}, \dots, m_{nn}) X^{n-1} + \dots + c_0(m_{11}, \dots, m_{nn})$

$c_0(M), \dots, c_{n-1}(M) \in R_{\mathbb{C}}^G$

G-invariant poly on $\mathfrak{g}^*_{\mathbb{C}}$.

Claim: $R_{\mathbb{C}}^G \cong \mathbb{C}[c_0, \dots, c_{n-1}]$

(Chevalley's restriction thm)

Thus $[\mathfrak{g}^*_{\mathbb{C}}] \cong \mathbb{C}^{n-1}$

$\mathfrak{g}^*_{\mathbb{C}} \twoheadrightarrow [\mathfrak{g}^*_{\mathbb{C}}] \cong \mathbb{C}^n$ surj map.

$M \mapsto (c_0(M), \dots, c_{n-1}(M))$

$G = PGL_n(\mathbb{R})$, $W \cong C_2$, sends $X \mapsto X$.
~~Proofs~~ $\det(X) = \det(-X)$. Hence $\det \in \mathbb{C}[H_{\mathbb{C}}]^{W_{\mathbb{C}}}$

<u>Scaling & conjugating</u>

$$[\mathfrak{g}^*_{\mathbb{C}}] = \mathrm{Hom}_{\mathbb{C}}(R_{\mathbb{C}}^{G}, \mathbb{C})$$

$$\phi \in [\mathfrak{g}^*_{\mathbb{C}}], \qquad \phi : R_{\mathbb{C}}^{G} \to \mathbb{C}$$

Operations :> Scaling : $\lambda\phi : R_{\mathbb{C}}^{G} \to \mathbb{C}$

$$x \mapsto \lambda\phi(x)$$

> Conjugating : $\bar{\phi} : R_{\mathbb{C}}^{G} \to \mathbb{C}$

$$x \mapsto \overline{\phi(x)}$$

$G = PGL_2(\mathbb{R})$, $R_{\mathbb{C}}^{G} \cong \mathbb{C}[\det]$
$[\mathfrak{g}^*_{\mathbb{C}}] \cong \mathbb{C}$

Scaling & Conjugating are just the usual operations on $\mathbb{C}$.

<u>How $[i\mathfrak{g}^*] \hookrightarrow [\mathfrak{g}^*_{\mathbb{C}}]$</u>

$$[i\mathfrak{g}^*] := \{\lambda \in [\mathfrak{g}^*_{\mathbb{C}}] \mid \lambda = -\bar{\lambda}\}, \quad i\mathfrak{g}^* \to [i\mathfrak{g}^*]$$

~~$\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}_2(\mathbb{C})$~~

$$\mathfrak{g} \twoheadrightarrow [i\mathfrak{g}^*]$$

~~since M & iM share same det in $\mathfrak{sl}_n(\mathbb{C})$ (up to ± 1)~~

$$M \mapsto [iM]$$

since $-\overline{iM} = iM$ for M real matrix

GIT quotients → Irred rep

Case of $PGL_2(\mathbb{R}) = G$

π irred rep of $PGL_2(\mathbb{R})$

$$Z(\mathcal{U}(\mathfrak{g})) = \mathbb{C}[\Omega_c]$$

$$-4\Omega_c = A^2 + B^2 - C^2$$
$$= A^2 + 2A + 4fe$$

Root space decomp : $\mathfrak{g}_{\mathbb{C}} = \underbrace{n^+}_{sp\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}} \oplus \underbrace{t_{\mathbb{C}}}_{sp(A)} \oplus \underbrace{n^-}_{sp\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}$

e $\quad$ f

Weyl grp: $W_{\mathbb{C}} = \{e, w\}$. acts on $t_{\mathbb{C}} = \mathbb{C}\, sp(A)$

$w : A \mapsto -A$

$e : A \mapsto A$.

$a_{\mathbb{C}}^* \to \mathfrak{g}_{\mathbb{C}}^*$ (extend trivially)

$$-4\Omega_c = \underbrace{A^2 + 2A}_{\in\, \mathcal{U}(t_{\mathbb{C}})} + \underbrace{4fe}_{n^- n^+}$$

$$\left\{ \mathcal{U}(\mathfrak{g}_{\mathbb{C}}) = \mathcal{U}(t_{\mathbb{C}}) \oplus n_{\mathbb{C}}^- \mathcal{U}(\mathfrak{g}_{\mathbb{C}}) \oplus \mathcal{U}(\mathfrak{g}_{\mathbb{C}})\, n_{\mathbb{C}}^+ \right\}$$

If $\lambda_\pi = [cA^*]$ $\cancel{\ }c^*$

$\Rightarrow \quad \lambda_\pi(A^2-1) = c^2-1$

\# Principal series: $\pi = \rho(iT)$

Ω_c acts by $\lambda = 1/4 - \cancel{\ }(iT)^2$
$= 1/4 + T^2$

$\lambda_\pi \in [\mathfrak{g}^*_c]$ given by cA^*

Thus, $c^2 - 1 = -4\lambda = -4T^2 - 1$

$\cancel{\ } \Rightarrow \quad c = \pm 2iT$

Thus $\underset{T>0}{\rho(iT)} \longleftrightarrow 2iTA^* \in [i\mathfrak{g}^*] \cong [\mathfrak{g}^\sim]$

Real For.

$\left(\begin{array}{l}\text{By trace pairing,}\\ 2iTA^* \leftrightarrow \cancel{\ }iTA\end{array}\right)$

\# Discrete series: $c \cancel{\ } = \frac{1}{2}(k-1)$

$\lambda_\pi = (k-1)A^*$

$(k-1)A^* \mapsto \frac{k-1}{2}A \quad \det \frac{-(k-1)^2}{4}$

Matches det of $\frac{1}{2}(k-1)iC$.

$D^{\pm}(\cancel{\ }k) \longleftrightarrow \cancel{\ }\ i(k-1)C^* \qquad (\leftrightarrow \frac{k-1}{2}iC)$

Eigenvalues: $[\mathfrak{g}^*_{\mathbb{C}}] \cong \mathbb{C}[\mathfrak{t}^*]/W$

$$\lambda \mapsto t_\lambda \leftrightarrow \{\lambda_1, \dots, \lambda_n\}$$

eigenvalues of t_λ

$$\lambda \in [\mathfrak{g}^*_{\mathbb{C}}] \to P_\lambda(\lambda) := X^n + c_{n-1}(\lambda) X^{n-1} + \dots + c_n(\lambda)$$

Monic poly of deg n over $\mathbb{C}$.

Char poly P_λ = Char poly $(t_\lambda) = (X - \lambda_1) \cdots (X - \lambda_n)$

$$\underset{\substack{\cong \\ [i\mathfrak{g}^*]}}{\lambda \in [\mathfrak{g}^{\wedge}]} \ (\subseteq [\mathfrak{g}^*_{\mathbb{C}}]) \iff -\bar{t}_\lambda \in W t_\lambda$$

$$\iff c_1(\lambda), c_3(\lambda), \dots \in i\mathbb{R}$$

$$c_2(\lambda), c_4(\lambda), \dots \in \mathbb{R}.$$

Scaling operation scales the eigenvalues.

$$(\lambda_1, \dots, \lambda_n) \mapsto (T\lambda_1, \dots, T\lambda_n)$$

Satake par: π irred rep $\longrightarrow \lambda_\pi \in [\mathfrak{g}^*_{\mathbb{C}}]$

$$\downarrow$$

eigval $\{\lambda_{\pi,1}, \dots, \lambda_{\pi,n}\}$

Archimedean Satake par.

$$L(\pi, s) = \prod_j \Gamma_{\mathbb{R}}(s + \underbrace{\lambda_{\pi,j}^{\tau}}_{\text{Langlands par.}} + a_j)$$

$a_j \in \{0, 1\}$

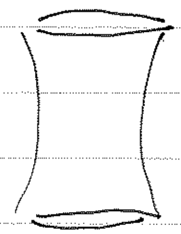
$(x + iy)^\tau := |x| + iy$

Kirillov's orbit method.

Symplectic measure on $\mathcal{O}_\pi$

Recall $\quad \pi = \mathcal{P}(iT)$ Principal series, Δ-eigenvalue $\lambda = 1/4 + T^2$

$$\mathcal{O}_\pi = \{(a,b,c) \in \mathbb{R}^3 \mid a^2 + b^2 - c^2 = T^2\}$$



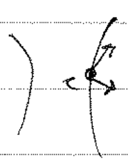
$\mathcal{O}_{TA}$ orbit in $\mathfrak{g}^*$

Parametrised by (ϕ, v) $\quad \phi \in [0, 2\pi)$, $v \in \mathbb{R}$

$$\tau(\phi, v) = T(\cosh v \cos\phi, \cosh v \sin\phi, \sinh v)$$

Symplectic measure: Measure on $\overset{\text{orbit}}{\mathcal{O}}$ defined on tangent sp.
(naturally defined)

$\tau \in \mathcal{O} \subseteq \mathfrak{g}^*$



Tangent sp.

Tangent space $T_\tau \mathcal{O}$ spanned by $ad_X^* \tau$ $\quad X \in \mathfrak{g}$

$$\gamma(t) = Ad_{\exp(tB)} \tau$$
$$\gamma'(0) \leftrightarrow ad_B^* \tau$$

$\left\{ ad_{X^*} \tau = [X, X_\tau] \right.$ when $\tau \leftrightarrow X_\tau \in \mathfrak{g}$ ~~by trace pairing~~

Measure: $G_\tau : T_\tau \mathcal{O} \times T_\tau \mathcal{O} \to \mathbb{R}$

$$(ad_X^*, ad_Y^*) \to \langle \tau, [X, Y] \rangle$$

Symplectic measure $= \omega_{\mathcal{O}} = 6/4\pi$

Kirillov: $\left(\chi_\pi \overset{\pi \text{ tempered, unitary.}}{\rightsquigarrow} \text{Integral in } \mathcal{O}_\pi \right)$

Thm (Rossmann) π tempered unitary irrep of G, reductive grp

$\lambda_\pi \in [\mathfrak{g}^\wedge]$ infinitesimal character.

$\exists$ unique finite union of coadj orbits $\mathcal{O}_\pi$

made up of regular elements of $\mathfrak{g}^\wedge$

contained in preimage of λ_π

st

(i) $$\chi_\pi(\exp(X)) = \frac{1}{\sqrt{j(x)}} \int_{\xi \in \mathcal{O}_\pi} \underbrace{e^{\langle \xi, X \rangle} \omega_{\mathcal{O}_\pi}(\xi)}_{\text{FT of symplectic measure.}}$$

$X \subseteq \mathfrak{g}$ small nbh of 0.

$\left\{ \begin{array}{l} j(x) \text{ Jacobian of } \exp : \mathfrak{g} \to G \text{ map.} \\ \text{Regular elements: (Coadj orbit has maximal dimension.)} \\ \quad \text{Only } 0 \text{ is not regular here..} \end{array} \right.$

(ii) Infinitesimal char of $\mathcal{O}_\pi$ = Infinitesimal char of π

((iii) If infinitesimal char of π is regular, $\mathcal{O}_\pi$ is a co-adj orbit.)

$$\mathrm{Op}_h(a) \cdot \mathrm{Op}_h(a) = \mathrm{Op}_h(a *_h a)$$

$a *_h a = a$ when $r > T^{-1/2}$

So bump can't be too small.

Thus $\mathrm{Op}_h(a)$ picks out a family of proj. operator.

(ii) Size of family $= \mathrm{tr}(\mathrm{Op}_h(a))$

$=$ Kirillov's formula $= \mathrm{vol}(\mathrm{supp}(a))$

(iii) Understanding image of $\mathrm{Op}_h(a)$

(Image = Microlocalised vectors at τ

Radius r = Extent of microlocalisation

v microloc at τ of order δ if

$\forall X \in \mathfrak{g}, \quad X \cdot v = \langle \tau, X \rangle v + O(\delta \|v\|)$)

If v microlocal at τ

$$\Rightarrow \quad \mathrm{Op}_h(a) v \approx \int_{X \in \mathfrak{g}} \chi \, a_h^\vee(X) \, \pi(e^X) v \, dX$$

ignore χ

$$= \int_{X \in \mathfrak{g}} \underbrace{a_h^\vee(X) \, e^{\langle \tau, X \rangle}}_{= \text{FT of } a_h^\vee} v \, dX$$

$$= a_h(\tau) v$$

Thus,

v microloc at τ $\Leftrightarrow$ $v \in \mathrm{Im}\, \mathrm{Op}_h(a)$