# Proof Techniques – Putnam Seminar 2026

9/29/26

## 1 Proof Techniques

- Contradiction: Assume the statement is false, then show contradiction to the hypothesis or to a true statement.
  - Example: Prove that there are infinitely many primes that have a remainder of 1 after dividing by 4.
- Induction: Assuming a statement is true for $1 \leq i \leq n$, show that the statement holds for $n+1$, and show that it holds for base cases.
- Pigeonhole principle: If $kn+1$ objects are distributed among $n$ boxes, then one box must contain at least $k+1$ objects.
- Invariants and monovariants: Find some property of the problem that remains constant (invariant) or is only increasing/decreasing (monovariant).
  - Example: Given the integers 1 to 100 on the whiteboard, we erase any two integers $a, b$ and replace them with $ab+a+b$. Prove that the final number on the whiteboard is always the same, and compute that number.

## 2 Problems

1. Suppose that there are $2n+1$ airports, where $n$ is a positive integer. The distances between any two airports are all different. For each airport, exactly one airplane departs from it and is destined for the closest airport. Prove that there is an airport which has no airplanes destined for it.
2. There is a heap of 1001 stones on a table. You are allowed to perform the following operation: you choose one of the heaps containing more than one stone, throw away a stone from the heap, then divide it into two smaller (not necessarily equal) heaps. Is it possible to reach a situation in which all the heaps on the table contain exactly 3 stones by performing the operation finitely many times?
3. You play a game with a single stack of $n$ coins. On each of your turns, you select one of your stacks of coins (that has at least two coins) and split it into two stacks, each with at least one coin. Your score for that turn is the product of the sizes of the two resulting stacks (for example, if you split a stack of 5 coins into a stack of 3 coins and a stack of 2 coins, your score would be $3 \cdot 2 = 6$). You continue taking turns until all your stacks have only one coin in them. What is the maximal possible total score?
4. Every point of three-dimensional space is colored red, green, or blue. Prove that one of the colors attains all distances, meaning that any positive real number represents the distance between two points of this color.
5. The points of the plane are colored by finitely many colors. Prove that one can find a rectangle with vertices of the same color.
6. Prove the AM-GM inequality by induction: For positive real numbers $a_1, \ldots, a_n$, we have that

$$\frac{a_1 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \cdots a_n}.$$

(Hint: Show that if AM-GM holds for $n$ variables, it holds for $2n$ variables, and then show that if it holds for $n$ variables, it holds for $n-1$ variables.)

7. (2013 Putnam A1) Recall that a regular icosahedron is a convex polyhedron having 12 vertices and 20 faces; the faces are congruent equilateral triangles. On each face of a regular icosahedron is written a nonnegative integer such that the sum of all 20 integers is 39. Show that there are two faces that share a vertex and have the same integer written on them.

8. (2022 Putnam B3) Assign to each positive real number a color, either red or blue. Let $D$ be the set of all distances $d > 0$ such that there are two points of the same color at distance $d$ apart. Recolor the positive reals so that the numbers in $D$ are red and the numbers not in $D$ are blue. If we iterate this recoloring process, will we always end up with all the numbers red after a finite number of steps?

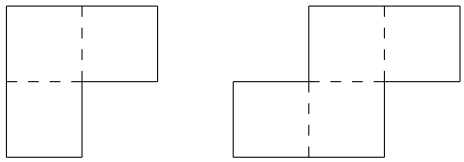
9. (2008 Putnam A3) Start with a finite sequence $a_1, a_2, \ldots, a_n$ of positive integers. If possible, choose two indices $j < k$ such that $a_j$ does not divide $a_k$, and replace $a_j$ and $a_k$ by $\gcd(a_j, a_k)$ and $\operatorname{lcm}(a_j, a_k)$, respectively. Prove that if this process is repeated, it must eventually stop and the final sequence does not depend on the choices made. (Note: gcd means greatest common divisor and lcm means least common multiple.) Start with a finite sequence $a_1, a_2, \ldots, a_n$ of positive integers.

10. (2006 Putnam B2) Prove that, for every set $X = \{x_1, x_2, \ldots, x_n\}$ of $n$ real numbers, there exists a non-empty subset $S$ of $X$ and an integer $m$ such that

$$\left| m + \sum_{s \in S} s \right| \leq \frac{1}{n+1}.$$

11. (2010 Putnam B3) There are 2010 boxes labeled $B_1, B_2, \ldots, B_{2010}$, and $2010n$ balls have been distributed among them, for some positive integer $n$. You may redistribute the balls by a sequence of moves, each of which consists of choosing an $i$ and moving *exactly* $i$ balls from box $B_i$ into any one other box. For which values of $n$ is it possible to reach the distribution with exactly $n$ balls in each box, regardless of the initial distribution of balls?

12. (2016 Putnam A4) Consider a $(2m-1) \times (2n-1)$ rectangular region, where $m$ and $n$ are integers such that $m, n \geq 4$. This region is to be tiled using tiles of the two types shown:



(The dotted lines divide the tiles into $1 \times 1$ squares.) The tiles may be rotated and reflected, as long as their sides are parallel to the sides of the rectangular region. They must all fit within the region, and they must cover it completely without overlapping.

What is the minimum number of tiles required to tile the region?