Putnam Seminar - Sequences and Series

October 2026

1 Tools

Convergence

Definition of Convergence of a Sequence

A real or complex sequence (a_n) converges to L if

$$\forall \varepsilon > 0 \; \exists N \in \mathbb{N} \; \forall n \geq N, \qquad |a_n - L| < \varepsilon.$$

Cauchy Criterion

A real or complex sequence (a_n) converges if and only if

$$\forall \varepsilon > 0 \; \exists N \in \mathbb{N} \; \forall m, n \geq N, \qquad |a_m - a_n| < \varepsilon.$$

A series $\sum_{n=1}^{\infty} a_n$ converges if and only if

$$\forall \varepsilon > 0 \; \exists N \in \mathbb{N} \; \forall m \geq n \geq N, \qquad \left| \sum_{k=n}^{m} a_k \right| < \varepsilon.$$

Integral Test

Let $f : [1, \infty) \to [0, \infty)$ be continuous and nonincreasing. Then

$$\sum_{n=1}^{\infty} f(n) \text{ converges} \quad \iff \quad \int_1^{\infty} f(x)\, dx < \infty.$$

Stolz–Cesàro Theorem

Let (a_n) and (b_n) be real sequences, with (b_n) strictly increasing and $b_n \to +\infty$. If

$$\lim_{n \to \infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = L,$$

then

$$\lim_{n \to \infty} \frac{a_n}{b_n} = L.$$

Here L may be finite or equal to $\pm\infty$.

2 Problems

1. Show that $\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}+\cdots}}}$ converges and find its limit.

2. Let $0 < a < b$ be real numbers. Define the sequences $(a_n)_n$ and $(b_n)_n$ by $a_0 = a$, $b_0 = b$ and

$$a_{n+1} = \sqrt{a_n b_n}, b_{n+1} = \frac{a_n + b_n}{2}, n \geq 0$$

. Prove that the two sequences are convergent and have the same limit.

3. Does $\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$ converge?

4. Prove that if $\sum a_n$ converges, where $(a_n)_n$ is a decreasing subsequence, then $\lim_{n\to\infty} na_n = 0$

5. Suppose that $(x_n)_n$ is a sequence of real numbers satisfying

$$x_{n+1} \leq x_n + \frac{1}{n^2}, \text{ for all } n \geq 1.$$

Prove that $\lim_{n\to\infty} x_n$ exists

6. Let p be a real number, $p \neq 1$. Compute

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} i^p}{n^{p+1}}$$

7. Let $a_0 = \pi/2$, and let $a_n = \sin(a_{n-1})$ for $n \geq 1$. Determine whether

$$\sum_{n=1}^{\infty} a_n^2$$

converges. (2020 Putnam A3)

8. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{if } x \leq e \\ xf(\ln x) & \text{if } x > e. \end{cases}$$

Does $\sum_{n=1}^{\infty} \frac{1}{f(n)}$ converge? (2008 Putnam A4)

9. Suppose that $a_0 = 1$ and that $a_{n+1} = a_n + e^{-a_n}$ for $n = 0, 1, 2, \ldots$. Does $a_n - \log n$ have a finite limit as $n \to \infty$? (Here $\log n = \log_e n = \ln n$.) (2012 Putnam B4)