Putnam (10.6)
Probability

1. Is it possible to state a protocol which only uses tosses of a fair coin, which is successful with probability exactly $1/3$?

2. Devise a way to place 6 (not necessarily distinct) positive integers on two dice, such that when they are rolled, the distribution of their sum is equal to the distribution of the sum of two ordinary dice. The two dice do not need to have the same labeling of numbers, and numbers can repeat on the same die.

3. There are n houses on a street, numbered $1, 2, \ldots n$. Each house is supposed to receive one piece of mail today, but unfortunately each piece is randomly delivered to a distinct house, so that the mapping from mail to houses is a uniformly random permutation of $1, 2, \ldots, n$. What is the expected number of houses that receive the correct mail?

4. You have a score which starts at 0, and every second your score increases by a uniformly random real number between 0 and 1. What is the expected amount of time it takes for your sum to surpass 1?

5. You start with m coins, and I start with n coins. Each round, each of us takes one of our coins, and we toss them. If our tosses match, I get both coins, and if our tosses are different, you get both coins. What is the expected number of tosses until one of us has no coins left?

6. Show that for any protocol of tossing a fair coin to achieve a success probability of exactly $1/3$, the expected number of tosses is at least 2.

7. What is the limiting probability (as $n \to \infty$) that a uniformly random permutation $(a_1, a_2, a_3, \ldots, a_n)$ has no adjacent pair of consecutive numbers (increasing or decreasing)? For example, in the permutation $(1, 5, 2, 4, 3)$, the 4 and 3 are an adjacent pair of consecutive numbers, and the permutation $(1, 3, 5, 2, 4)$ has no adjacent pair of consecutive numbers.

9. 1958 A3: A sequence of numbers $\alpha_i \in [0, 1]$ is chosen at random. Show that the expected value of n, where $\sum_1^n \alpha_i > 1$, $\sum_1^{n-1} \alpha_i \leq 1$ is e.

10. 2014 A4: Suppose X is a random variable that takes on only non-negative integer values, with E [X] = 1, E X2 = 2, and E X3 = 5. (Here E [y] denotes the expectation of the random variable Y .) Determine the smallest possible value of the probability of the event X= 0.

11. 2022 A4: Suppose that X1, X2, . . . are real numbers between 0 and 1 that are chosen independently and uniformly at ran- dom. Let S= $\sum$k i=1 Xi/2i, where k is the least positive integer such that Xk < Xk+1, or k= ∞ if there is no such integer. Find the expected value of S.

12. 2022 B5: For $0 \leq p \leq 1/2$, let $X_1, X_2, \ldots$ be independent random variables such that

$$X_i = \begin{cases} 1 & \text{with probability } p, \\ -1 & \text{with probability } p, \\ 0 & \text{with probability } 1-2p, \end{cases}$$

for all $i \geq 1$. Given a positive integer n and integers $b, a_1, \ldots, a_n$, let $P(b, a_1, \ldots, a_n)$ denote the probability that $a_1X_1 + \cdots + a_nX_n = b$. For which values of p is it the case that

$$P(0, a_1, \ldots, a_n) \geq P(b, a_1, \ldots, a_n)$$

for all positive integers n and all integers $b, a_1, \ldots, a_n$?