11/10/26
Putnam

Linear Algebra

1 Well-known statements

Integer matrices. A square matrix with all-integer entries has inverse consisting of all-integer entries if and only if its determinant is ± 1.

Area. If a triangle in the plane has coordinates (x_1, y_1), (x_2, y_2), and (x_3, y_3), then its area is the absolute value of:

$$\frac{1}{2} \cdot \det \begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix}.$$

Spectral mapping theorem. If an $n \times n$ square matrix A has eigenvalues $\lambda_1, \ldots, \lambda_n$ (possibly with multiplicity), and $P(x)$ is a polynomial, then the eigenvalues of the matrix $P(A)$ are $P(\lambda_1), \ldots, P(\lambda_n)$.

Commuting, sort of. For an $n \times n$ matrix A, let $\phi_k(A)$ denote the degree-k symmetric polynomial in the eigenvalues $\lambda_1, \ldots, \lambda_n$ of A:

$$\phi_k(A) = \sum_{i_1, i_2, \ldots, i_k} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}.$$

For example, $\phi_1(A)$ is the trace of A, and $\phi_n(A)$ is the determinant of A. Prove that for every $1 \leq k \leq n$, and every pair of $n \times n$ matrices A and B,

$$\phi_k(AB) = \phi_k(BA).$$

2 Problems

1. For any $n \times n$ matrix A with real entries,
$$\det(I_n + A^2) \geq 0.$$

2. Let A, B, and D be $n \times n$ matrices. Prove that
$$\det \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} = \det(AD).$$

3. Let A, B, C, and D be $n \times n$ matrices such that $AC = CA$. Prove that
$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AD - CB).$$

4. Let X and Y be $n \times n$ matrices, and let I_n be the $n \times n$ identity matrix. Prove that
$$\det(I_n - XY) = \det(I_n - YX).$$

5. There are given $2n + 1$ real numbers, $n \geq 1$, with the property that whenever one of them is removed, the remaining $2n$ can be split into two sets of n elements that have the same sum of elements. Prove that all the numbers are equal.

6. Let A be an $n \times n$ matrices such that a_{ij} is the entry in the i-th row and j-th column. Suppose that for every row i, $\sum_{j=1}^{n} |a_{ij}| < 1$. Prove that $I_n - A$ is invertible.

7. Let A be an $n \times n$ matrix. Prove that there exists an $n \times n$ matrix B such that $ABA = A$.

8. Let $k < n$ be two positive integers. Compute:
$$\det \begin{pmatrix} \binom{n}{0} & \binom{n}{1} & \cdots & \binom{n}{k} \\ \binom{n+1}{0} & \binom{n+1}{1} & \cdots & \binom{n+1}{k} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{n+k}{0} & \binom{n+k}{1} & \cdots & \binom{n+k}{k} \end{pmatrix}$$

9. Given distinct integers $x_1, x_2, \ldots x_n$, prove that $\prod_{i<j}(x_i - x_j)$ is divisible by $1!2! \cdots (n-1)!$.

2003 B1: Do there exist polynomials a(x), b(x), c(y), d(y) such that $1 + xy + x^2y^2 = a(x)c(y) + b(x)d(y)$ holds identically?

2008 A2: Alan and Barbara play a game in which they take turns filling entries of an initially empty 2008 $\times$2008 array. Alan plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when all the entries are filled. Alan wins if the determinant of the resulting matrix is nonzero; Barbara wins if it is zero. Which player has a winning strategy?

2009 B3: Let Q be an n-by-n real orthogonal matrix, and let $u \in R^n$ be a unit column vector (that is, $u^T u = 1$). Let $P = I - 2uu^T$, where I is the n-by-n identity matrix. Show that if 1 is not an eigenvalue of Q, then 1 is an eigenvalue of PQ.

2009 B4: Say that a polynomial with real coefficients in two variables, x, y, is balanced if the average value of the polynomial on each circle centered at the origin is 0. The balanced polynomials of degree at most 2009 form a vector space V over R. Find the dimension of V.