§1 Determinants

Some useful properties of the determinant:

- Exchanging two rows multiplies the determinant by -1.
- Multiplying a row by a scalar c multiplies the determinant by c.
- Adding a multiple of one row to another row does not change the determinant.
- For two square matrices A, B of the same size, $\det(AB) = \det(A)\det(B)$.

The above discussion also holds with "row" replaced by "column."

Problem 1.1. For what value of x does the following determinant have the value 2025?

$$\begin{vmatrix} 5+x & 8 & 2+x \\ 1 & 1+x & 3 \\ 2 & 1 & 2 \end{vmatrix}$$

Problem 1.2 (Geometric interpretation of 2×2 determinants)**.** Prove that the area of a parallelogram with vertices $(0,0)$, (a,c), $(a+b, c+d)$, and (b,d) is $|ad-bc|$.

Problem 1.3. Show that the determinant

$$\begin{vmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{vmatrix}$$

is nonnegative if all entries are real.

Example 1.4 (Vandermonde matrix)

Compute the determinant of the matrix

$$\begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{pmatrix}.$$

Problem 1.5. Compute the determinant of the Vandermonde matrix from Example 1.4 with the first row reversed.

Problem 1.6 (2014 A2)**.** Let A be the $n \times n$ matrix whose entry in the i-th row and j-th column is $\frac{1}{\min(i,j)}$ for $1 \le i, j \le n$. Compute $\det(A)$.

§ 2 Path Counting

Example 2.1

A walks from $(0,0)$ to $(3,3)$ in the coordinate plane only taking unit steps up or right. Similarly, B walks from $(-2,2)$ to $(1,5)$. How many ways can they walk such that their paths do not intersect?

The key is that the number of pairs of intersecting paths is the number of ways to walk to the other person's endpoint by swapping paths the first time they intersect.

Theorem 2.2 (Lindström–Gessel–Viennot)

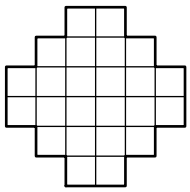
Consider an acyclic directed graph with distinct vertices $a_1, \dots, a_n$ and $b_1, \dots, b_n$. Suppose for all $i < j$ and $k < \ell$ that every path connecting a_i to b_ℓ intersects every path connecting a_j to b_k. Then the number of tuples $(p_1, \dots, p_n)$ of nonintersecting paths where p_i is a path from a_i to b_i is the determinant of the matrix

$$\begin{pmatrix} e(a_1,b_1) & e(a_1,b_2) & \cdots & e(a_1,b_n) \\ e(a_2,b_1) & e(a_2,b_2) & \cdots & e(a_2,b_n) \\ \vdots & \vdots & \ddots & \vdots \\ e(a_n,b_1) & e(a_n,b_2) & \cdots & e(a_n,b_n) \end{pmatrix}$$

where $e(a_i, b_j)$ is the number of paths from a_i to b_j.

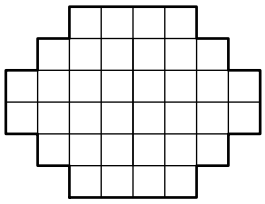
Example 2.3

Compute the number of ways to tile the following shape with dominoes.

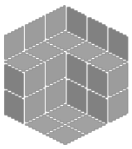


This shape is the Aztec *diamond* of order 3. In general, the number of domino tilings of the Aztec diamond of order n is $2^{n(n+1)/2}$.

Problem 2.4. Find the number of tilings of an order n Aztec diamond with the top and bottom rows cut off.



Problem 2.5. An AoPS logo is a tiling of a regular hexagon of side length 3 with 60° rhombuses of side length 1. How many AoPS logos are there?



§ 3 Additional Problems

Problem 3.1. Do there exist square matrices A and B such that $AB-BA = I$?

Problem 3.2 (2022 B2)**.** Let $\times$ represent the cross product in $\mathbb{R}^3$. For what positive integers n does there exist a set $S \subset \mathbb{R}^3$ with exactly n elements such that

$$S = \{v \times w : v, w \in S\}?$$

Problem 3.3 (2023 A4)**.** Let $v_1, \ldots, v_{12}$ be unit vectors in $\mathbb{R}^3$ from the origin to the vertices of a regular icosahedron. Show that for every vector $v \in \mathbb{R}^3$ and every $\varepsilon > 0$, there exist integers $a_1, \ldots, a_{12}$ such that $\|a_1v_1 + \cdots + a_{12}v_{12} - v\| < \varepsilon$.