§ 1 Invariants and Monovariants

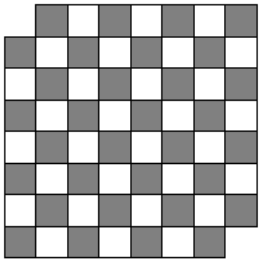
Some problems involve a process in which an operation is repeatedly performed.

Definition 1.1

An **invariant** is any quantity that remains unchanged throughout the process. A **monovariant** is any quantity that always decreases (or increases) throughout the process.

Oftentimes problems require finding the right invariant/monovariant.

Exercise 1.2. Two tiles are removed from opposite corners of a chessboard, as shown. Is it possible to tile the remaining tiles of the chessboard with 31 dominoes?



Exercise 1.3. Soldiers are facing either east or west in a line. At each step, if two directly face each other, they both turn around to face the opposite direction. Prove that this process terminates after a finite number of steps.

Exercise 1.4. The numbers $1, 2, \ldots, 100$ are written on the board. Joel picks any two numbers a, b on the board, erases them, and writes the number $|a - b|$ on the board. This process is repeated until only one number is remaining on the board. Is it possible for the number to be odd?

§ 2 Famous Games

In two-player games, a common technique is to use *symmetry*, by mirroring an opponent's moves.

Exercise 2.1. Alice and Bob take turns placing dominoes on a 1×20 board, with Alice going first. The dominoes may not overlap, and the last person to place a domino wins. Show that Alice has a winning strategy.

Exercise 2.2 (Nim)**.** Two players start with a pile of n stones. On each player's turn, they remove either one or two stones. The player who takes the last stone wins. Which player has a winning strategy?

§ 3 Problems

Problem 3.1. The numbers $1, 2, \ldots, 100$ are written on a blackboard. You may choose any two numbers a and b and erase them, replacing them with the single number $a + b - 1$. After 99 steps, only a single number will be left. What is it?

Problem 3.2. An 8×8 chessboard is colored in the usual way, but that's boring, so you decide to fix this. You can take any row, column, or 2×2 square, and reverse the colors inside it, switching black to white and white to black. Prove that it's impossible to end up with 63 white squares and 1 black square.

Problem 3.3. Ron and Martin are playing a game with a bowl containing 39 marbles. Each player takes turns removing 1, 2, 3 or 4 marbles from the bowl. The person who removes the last marble loses. If Ron takes the first turn to start the game, how many marbles should he remove to guarantee he is the winner?

Problem 3.4. A room is initially empty. Every minute, either two people enter or one person leaves. After exactly 3^{3^3} minutes, could the room contain exactly $3^{3^3}+1$ people?

Problem 3.5. A herd of 100 cows is divided into four pens: 10 cows in the north pen, 20 cows in the east pen, 30 cows in the south pen, and 40 cows in the west pen. The pens are connected through a gateway we can use to let three cows out of one pen and distribute them between the others. For instance, if we let three cows out of the south pen, we end up with 11 cows in the north pen, 21 cows in the east pen, 27 cows in the south pen, and 41 cows in the west pen.

Prove that we can never use this gateway to split the herd into four equal groups, with 25 cows in each of the four pens.

Problem 3.6. Two players play a game with a pile of 13 beans. They take turns picking up some number of beans; on his or her turn, a player can pick up exactly 1, exactly 4, or exactly 6 beans. The player who takes the last bean wins. Which player (the first or the second to move) wins this game, and how?

Problem 3.7. Three people are standing in a circle and given a black or white hat with equal probability. They can see everyone else's hats but not their own. Simultaneously, everyone either guesses their own hat color or remains silent. They win if at least one guess was made, and all guesses made were correct. Find a strategy that guarantees that they win at least $\frac{3}{4}$ of the time.

Problem 3.8. Bela and Jenn play the following game on the closed interval $[0, n]$ of the real number line, where n is a fixed integer greater than 4. They take turns playing, with Bela going first. At his first turn, Bela chooses any real number in the interval $[0, n]$. Thereafter, the player whose turn it is chooses a real number that is more than one unit away from all numbers previously chosen by either player. A player unable to choose such a number loses. Using optimal strategy, which player will win the game? Justify your answer.

Problem 3.9. The game of double chess is played like regular chess, except each player makes two moves in their turn (white plays twice, then black plays twice, and so on). Show that white can always win or draw.

Problem 3.10. Alina and Cheuk Hei have an $m \times n$ chocolate bar, where m and n are positive integers. Alina moves first, and the players take turns breaking one of the pieces of the chocolate into two smaller pieces along the grid of the chocolate bar; the last player to make a move wins. For which (m, n) does Alina have a winning strategy?

Problem 3.11. Seven squares of an 8×8 grid are shaded. At each step, we shade in each unshaded square that has at least two shaded neighboring squares (horizontally or vertically). Prove that this process cannot end in the entire grid being shaded.

Exercise 3.12. There are 13 green, 15 blue, and 17 red chameleons on Camelot Island. Whenever two chameleons of different colors meet, they both swap to the third color (i.e., a green and blue would both become red). Is it possible for all chameleons to become one color?