PUTNAM (10/28)
Probability

2002-B-1. Shanille O'Keal shoots free throws on a basketball court. She hits the first and misses the second, and thereafter the probability that she hits the next shot is equal to the proportion of shots she has hit so far. What is the probability that she hits exactly 50 of her first 100 shots?

2001-A-2. You have coins $C1, C2, \cdots, Cn$. For each k, coin Ck is biased so that, when tossed, it has probability $1/(2k+1)$ of falling heads. If the n coins are tossed, what is the probability that the number of heads is odd? Express the answer as a rational function of n.

2007-A-3. Let k be a positive integer. Suppose that the integers $1,2,3,\cdots,3k+1$ are written down in random order. What is the probability that at no time during the process, the sum of the integers that have been written up to that time is a positive integer divisible by 3? Your answer should be in closed form, but may include factorials.

2014-A-4. Suppose X is a random variable that takes on only nonnegative integers values, with $E[X] = 1$, $E[X^2] = 2$, and $E[X^3] = 5$. (Here E[Y] denotes the expectation of the random variable Y.) Determine the smallest possible value of the probability of the event $X = 0$.

2017-A-5. Each of the integers from 1 to n is written on a separate card, and then the cards are combined into a deck and shuffled. Three players, A, B, and C, take turns in the order A,B,C,A,... choosing one card at random from the deck. (Each card in the deck is equally likely to be chosen.) After a card is chosen, that card and all higher-numbered cards are removed from the deck, and the remaining cards are reshuffled before the next turn. Play continues until one of the three players wins the game by drawing the card numbered 1. Show that for each of the three players, there are arbitrarily large values of n for which that player has the highest probability among the three players of winning the game.

2006-A-6. Four points are chosen uniformly and independently at random in the interior of a given circle. Find the probability that they are the vertices of a convex quadrilateral.

7. You write down two distinct real numbers, one on the left and one on the right. I decide whether to ask you to reveal the left number or the right number. After that, I guess which of your numbers was the larger one. Do I have a strategy for which my chance of winning is strictly greater than half?

9. Oh no! You're above a pit of fire and the only way out is a 1000-rung escape ladder. You're on the first rung. Every second, a standard (6 sided) die rolls. If it's a 1 or 2, you move down one rung. Anything 3 or greater, you move up one rung. If you move down from the first rung, you fall into the fire! If you move up from the 1000th rung, you escape. What is the probability that you escape, as a percent rounded to the nearest hundredth?

10. You throw a die until you get 6. What is the expected number of throws (including the throw giving 6), conditioned on the event that all throws gave even numbers?

11. Consider a game between two people, each of whom rolls a die. The rules are that the die must roll for a distance of at least 1 foot on the carpet. The larger number wins the round, but in the event that there is a tie, the roller whose die stopped moving first wins, with one exception: if both are 6's, then they re-roll. What is the optimal strategy?