Putnam - Polynomials - 10-21

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1 Things to know

1. Division Algorithm. $p(x) = g(x)q(x) + r(x)$, where $\deg(r(x)) < \deg(q(x))$
2. Vietas Formulas. $s_k = (-1)^k \frac{a_{n-k}}{a_n}$., where $s_k = \sum_{\{i_1, i_2, \ldots i_k\} \subseteq [n]} x_{i_1} x_{i_2} \ldots x_{i_k}$, and x_i is the ith root of the polynomial and a_i is the coefficient of the x^i term.

2 Easier Problems

1. Consider all lines which meet the graph of
$$y = 2x^4 + 7x^3 + 3x - 5$$
in four distinct points, say $(x_i, y_i),\ i = 1, 2, 3, 4$. Show that
$$\frac{x_1 + x_2 + x_3 + x_4}{4}$$
is independent of the line and find its value. [1977 Putnam A1]
2. What is the remainder when $X^{1982} + 1$ is divided by $X - 1$? Verify your answer. [1982 VTRMC #1]
3. Determine all polynomials $P(x)$ such that $P(x^2 + 1) = (P(x))^2 + 1$ and $P(0) = 0$. [1971 Putnam A2]
4. The product of two of the four roots of the quartic equation $x^4 - 18x^3 + kx^2 + 200x - 1984 = 0$ is -32. Determine the value of k. [1984 USAMO #1]

3 Medium Problems

1. Let a, b, c, d be real numbers such that $b - d \geq 5$ and all zeros x_1, x_2, x_3, and x_4 of the polynomial $P(x) = x^4 + ax^3 + bx^2 + cx + d$ are real. Find the smallest value the product $(x_1^2 + 1)(x_2^2 + 1)(x_3^2 + 1)(x_4^2 + 1)$ can take. [2014 USAMO #1]

2. Let a, b, and c denote three distinct integers, and let P denote a polynomial having all integral coefficients. Show that it is impossible that $P(a) = b$, $P(b) = c$, and $P(c) = a$. [1974 USAMO #1]

3. If $P(x)$ denotes a polynomial of degree n such that $P(k) = k/(k+1)$ for $k = 0, 1, 2, \ldots, n$, determine $P(n+1)$. [1975 USAMO #3]

4. Let n be an even positive integer. Let p be a monic, real polynomial of degree $2n$; that is to say, $p(x) = x^{2n} + a_{2n-1}x^{2n-1} + \cdots + a_1 x + a_0$ for some real coefficients $a_0, \ldots, a_{2n-1}$. Suppose that $p(1/k) = k^2$ for all integers k such that $1 \leq |k| \leq n$. Find all other real numbers x for which $p(1/x) = x^2$. [2023 Putnam A2]

4 Hard Problems

1. For which real polynomials p is there a real polynomial q such that

$$p(p(x)) - x = (p(x) - x)^2 q(x)$$

for all real x? [2024 Putnam A2]

2. Find all polynomials whose coefficients are equal either to 1 or -1 and whose zeros are all real. [Putnam and Beyond # 159] (Indian Olympiad Training Program 2005)

3. Find all pairs of polynomials $p(x)$ and $q(x)$ with real coefficients for which

$$p(x)q(x+1) - p(x+1)q(x) = 1.$$

[2010 Putnam B4]