1 Determinants

1. For what value of x does the following determinant have the value 2013?

$$\begin{vmatrix} 5+x & 8 & 2+x \\ 1 & 1+x & 3 \\ 2 & 1 & 2 \end{vmatrix}$$

2. Define the determinant $D_1 = |1|$, the determinant $D_2 = \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix}$, and the determinant $D_3 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 5 \end{vmatrix}$. In general, for each positive integer n, let the determinant D_n have 1s in every position of its first row and first column, 3s in the remaining positions of the second row and second column, 5s in the remaining positions of the third row and third column, and so forth. Find the least n so that $D_n \geq 2015$.

3. Alan and Barbara play a game in which they take turns filling entries of an initially empty 2008×2008 array. Alan plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when all the entries are filled. Alan wins if the determinant of the resulting matrix is nonzero; Barbara wins if it is zero. Which player has a winning strategy?

4. Show that the determinant:

$$\begin{vmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{vmatrix}$$

   is non-negative, if its elements a, b, c, etc., are real.

5. Prove the Vandermonde matrix identity, which states the following.

   Let $x_1, x_2, \ldots, x_n$ be scalars, and consider the Vandermonde matrix

$$V(x_1, \ldots, x_n) = \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{pmatrix}.$$

   Prove that its determinant has the closed form

$$\det V(x_1, \ldots, x_n) = \prod_{1 \leq i < j \leq n} (x_j - x_i).$$

6. Let d_n be the determinant of the $n \times n$ matrix whose entries, from left to right and then from top to bottom, are $\cos 1, \cos 2, \ldots, \cos n^2$. (For example,

$$d_3 = \begin{vmatrix} \cos 1 & \cos 2 & \cos 3 \\ \cos 4 & \cos 5 & \cos 6 \\ \cos 7 & \cos 8 & \cos 9 \end{vmatrix}.$$

The argument of cos is always in radians, not degrees.) Evaluate $\lim_{n\to\infty} d_n$.

7. In Determinant Tic-Tac-Toe, Player 1 enters a 1 in an empty 3×3 matrix. Player 0 counters with a 0 in a vacant position, and play continues in turn until the 3×3 matrix is completed with five 1's and four 0's. Player 0 wins if the determinant is 0 and player 1 wins otherwise. Assuming both players pursue optimal strategies, who will win and how?

8. Let A and B be 3×3 matrices with real elements such that $\det A = \det B = \det(A + B) = \det(A - B) = 0$. Prove that $\det(xA + yB) = 0$ for any real numbers x and y.

9. Let p be a prime number. Prove that the determinant of the matrix

$$\begin{pmatrix} x & y & z \\ x^p & y^p & z^p \\ x^{p^2} & y^{p^2} & z^{p^2} \end{pmatrix}$$

is congruent modulo p to a product of polynomials of the form $ax+by+cz$, where a, b, c are integers. (We say two integer polynomials are congruent modulo p if corresponding coefficients are congruent modulo p.)

2 Eigenvalues/Vectors

1. Suppose A is a real $n \times n$ matrix which satisfies $A^3 = A + I_n$. Show that A has a positive determinant.

2. Let Q be an n-by-n real orthogonal matrix, and let $u \in \mathbb{R}^n$ be a unit column vector (that is, $u^T u = 1$). Let $P = I - 2uu^T$, where I is the n-by-n identity matrix. Show that if 1 is not an eigenvalue of Q, then 1 is an eigenvalue of PQ.

3. For an integer $n \geq 3$, let $\theta = 2\pi/n$. Evaluate the determinant of the $n \times n$ matrix $I + A$, where I is the $n \times n$ identity matrix and $A = (a_{jk})$ has entries $a_{jk} = \cos(j\theta + k\theta)$ for all j, k.