PUTNAM (9/23)

Induction, Contradiction, & Other Proofs

1. Prove that $n! > 2^n$ for all $n \geq 4$.

2. Prove that for any integer $n \geq 1$, $2^{2n} - 1$ is divisible by 3.

3. Let a and b two distinct integers, and n any positive integer. Prove that $a^n - b^n$ is divisible by $a - b$.

4. The Fibonacci sequence $0, 1, 1, 2, 3, 5, 8, 13, \ldots$ is defined as a sequence whose two first terms are $F_0 = 0$, $F_1 = 1$ and each subsequent term is the sum of the two previous ones: $F_n = F_{n-1} + F_{n-2}$ (for $n \geq 2$). Prove that $F_n < 2^n$ for every $n \geq 0$.

5. Let a_n be the following expression with n nested radicals:

$$a_n = \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2 + \sqrt{2}}}}.$$

Prove that $a_n = 2 \cos \frac{\pi}{2^{n+1}}$.

6. Suppose n coins are given, named C1, . . . , Cn. For each k, Ck is biased so that, when tossed, it has probability 1/(2k + 1) of falling heads. If the n coins are tossed, what is the probability that the number of heads is odd? Express the answer as a rational function of n.

7. Seventeen people correspond by mail with one another — each one with all the rest. In either letter only three topics are discussed. Each pair of correspondents deals with only one of the topics. Prove that there are at least three people who write to each other about the same topic.

8.

Prove that any convex pentagon whose vertices (no three of which are collinear) have integer coordinates must have area ≥ 5/2.

9.

Let $a_0 = \pi/2$, and let $a_n = \sin(a_{n-1})$ for $n \geq 1$. Determine whether

$$\sum_{n=1}^{\infty} a_n^2$$

converges.

10.

Let T_n denote the collection of non-empty subsets of {1, …., n}, where each subset contains no consecutive integers. For each S ∈ T, let P_S be the square of the product of all elements of S. Show that the sum of P_S over all S ∈ T is (n+1)! - 1. For example, when n = 3, T = {{1}, {2}, {3}, {1,3}}, the values of P_S are 1, 4, 9, and 9, and 1 + 4 + 9 + 9 + 23 = 4! - 1.

11.

Consider a set of 1985 positive integers, not necessarily distinct, and none with prime factors bigger than 23. Prove that there must exist four integers in this set whose product is equal to the fourth power of an integer.

12.

Let a1, a2, . . . , an be distinct positive integers and let M be a set of n − 1 positive integers not containing s = a1 +· · · + an. A grasshopper is to jump along the real axis, starting at the point 0 and making n jumps to the right with lengths a1, a2, . . . , and in some order. Prove that the order can be chosen in such a way that the grasshopper never lands on any point in M.