Columbia Putnam Seminar

11/25/25

1 Problems

1. Let E be a set with n elements and F a set with p elements, with $p \leq n$. How many surjective (i.e., onto) functions $f : E \to F$ are there?

2. Prove that
$$\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}.$$

3. Denote by V the number of vertices of a convex polyhedron, and by Σ the sum of the (planar) angles of its faces. Prove that $2\pi V - \Sigma = 4\pi$.

4. Dexter is running a pyramid scheme. In Dexter's scheme, he hires *ambassadors* for his company, Lie Ultimate. Any ambassador for his company can recruit up to two more ambassadors (who are not already ambassadors), who can in turn recruit up to two more ambassadors each, and so on (Dexter is a special ambassador that can recruit as many ambassadors as he would like). An ambassador's *downline* consists of the people they recruited directly as well as the downlines of those people. An ambassador earns *executive status* if they recruit two new people and each of those people has at least 70 people in their downline (Dexter is *not* considered an executive). If there are 2020 ambassadors (including Dexter) at Lie Ultimate, what is the maximum number of ambassadors with executive status?

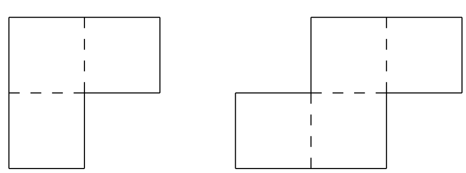
5. Let p be an odd prime number. Find the number of subsets of $\{1, \ldots, p\}$ with the sum of elements divisible by p.

6. (2020 A2) Let k be a nonnegative integer. Evaluate
$$\sum_{j=0}^{k} 2^{k-j} \binom{k+j}{j}.$$

7. (2020 B2) Let k and n be integers with $1 \leq k < n$. Alice and Bob play a game with k pegs in a line of n holes. At the beginning of the game, the pegs occupy the k leftmost holes. A legal move consists of moving a single peg to any vacant hole that is further to the right. The players alternate moves, with Alice playing first. The game ends when the pegs are in the k rightmost holes, so whoever is next to play cannot move and therefore loses. For what values of n and k does Alice have a winning strategy?

8. (2022 B3) Assign to each positive real number a color, either red or blue. Let D be the set of all distances $d > 0$ such that there are two points of the same color at distance d apart. Recolor the positive reals so that the numbers in D are red and the numbers not in D are blue. If we iterate this recoloring process, will we always end up with all the numbers red after a finite number of steps?

9. (2016 A4) Consider a $(2m-1) \times (2n-1)$ rectangular region, where m and n are integers such that $m, n \geq 4$. This region is to be tiled using tiles of the two types shown:



(The dotted lines divide the tiles into 1×1 squares.) The tiles may be rotated and reflected, as long as their sides are parallel to the sides of the rectangular region. They must all fit within the region, and they must cover it completely without overlapping.

What is the minimum number of tiles required to tile the region?

10. (2020 A5) Let a_n be the number of sets S of positive integers for which

$$\sum_{k \in S} F_k = n,$$

where the Fibonacci sequence $(F_k)_{k \geq 1}$ satisfies $F_{k+2} = F_{k+1} + F_k$ and begins $F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3$. Find the largest integer n such that $a_n = 2020$.

11. (2018 B6) Let S be the set of sequences of length 2018 whose terms are in the set $\{1, 2, 3, 4, 5, 6, 10\}$ and sum to 3860. Prove that the cardinality of S is at most

$$2^{3860} \cdot \left(\frac{2018}{2048}\right)^{2018}.$$

12. Twenty-one girls and twenty-one boys took part in a mathematics competition. It turned out that

    (a) each contestant solved at most six problems, and

    (b) for each pair of a girl and a boy, there was at least one problem that was solved by both the girl and the boy.

    Show that there is a problem that was solved by at least three girls and at least three boys.